No Perfect Squares are Perfect Numbers

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Background

- Proper divisor - The proper divisors of some positive integer *n* are all its factors excluding *n* itself.
- Perfect Number - A positive integer whose proper divisors sum to itself.
- 6, 28, 496, 8128 are the 4 smallest perfect numbers.
- Sum of proper divisors function - sums up all proper divisors.
- No known perfect numbers are perfect squares.

Euclid-Euler Theorem / History

- Mersenne Prime (named after the polymath Marin Mersenne) - prime of the form 2^n-1 such that *n* is prime
- Euclid proved that if 2^n-1 is a Mersenne prime, then 2^n-1(2^{n-1}) is perfect.
- Leonhard Euler proved all even perfect numbers take this form.
- Euclid-Euler Theorem shows no even perfect numbers are perfect squares (on next slide)

No Even Perfect Squares are Perfect Numbers

- Let's take an even square number *B*, and assume it's perfect.
- So, by the Euclid-Euler theorem, it must be of the form $(2^{n-1})*(2^n - 1)$ for prime *n*.
- Since 2^{n-1} is a power of 2, it must be even. So, it represents the highest power of 2 that divides *B*.
- Since 2^n - 1 is one less than a power of 2, it must be odd. So, it represents the result of dividing *B* by 2^{n-1}. It's also the largest odd divisor of *B*.

No Even Perfect Squares are Perfect Numbers cont.

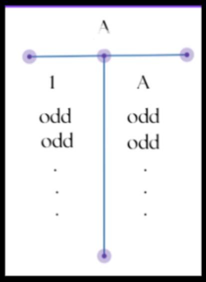
- Firstly, *B* can't be a power of 2, because if it was, then 2^n - 1 would equal 1, and then *n* would have to equal 1, which is impossible since 1 isn't prime.
- Therefore, *B* must have an odd divisor greater than 1, and since *B* is the product of two √*B*'s, √*B* must have an odd divisor greater than 1 (if it didn't, *B* wouldn't have one, either).
- Also, *B*'s odd divisor greater than 1 must be the square of √*B*'s odd divisor greater than 1.
- Therefore, *B*'s odd divisor greater than 1 is a perfect square, and therefore, composite. Since it's composite, it can't be a Mersenne prime.
- However, this is a contradiction. Therefore, no even square number can be perfect.

History cont.

- 52 known perfect numbers
- Unknown whether infinitely many perfect numbers exist
- The existence of odd perfect numbers is one of the oldest open math problems
- Euler has previously proven that no odd perfect numbers exist
- His proof uses multiplicative functions: functions such that f(a*b) = f(a) * f(b)

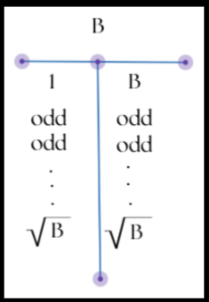
Proof

- Take an odd number *A* that is not a perfect square.
- In *A*'s factor tree, there will be some number of factor pairs. Each will consist of odd numbers, since *A* is odd and cannot have any even numbers as its factors, since if it did, it would have a 2 in its prime factorization, making it even.
- Since these odd numbers are in pairs, there will be an even number of factors in *A*.
- Since the definition of a perfect number is a number that is the sum of all its proper divisors (factors except itself), the factor that is *A* itself will be omitted, leaving an odd number of odd numbers.
- An odd number of odd numbers will add up to an odd number, which may very well be *A*.



Proof cont.

- However, take an odd perfect square *B*. Once again, there will be some factor pairs.
- By the definition of a perfect number, omit *B* from the numbers to be added up. However, since *B* is a perfect square, there will be two √*B*'s in the factor tree.
- We need to omit one of these from the numbers to be added.
- There will be an even number of odd numbers to add up left, since two from the pairs were omitted, which will sum to an even number.
- Since *B* is odd, this number cannot equal *B*.
- Therefore, no odd perfect squares are perfect numbers.



Suggestions for Further Reading

- Euler's *De Numeris Amicabilibus* (*On Amicable Numbers*)

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