Accessibility, Artistry, and Mathematics A Visual Proof of Heron's Result

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0. Abstract

In this undergraduate student research project, we apply Oliver Byrne's visual methodology to prove Heron of Alexandria's famous formula for triangular area. The proof, although rich and intriguing in its approach, requires a considerable amount of effort for the reader to fully understand and appreciate the result. We draw from the primary historical source for Heron's proof to create an original visual proof. Our goal in this project is to make Heron's proof more widely accessible to a range of readers, as well as to celebrate the artistry in Bynre's approach.

1. Introduction

Many geometric results and their proofs provide integral foundational context to a lot of math that students learn today, but can be difficult to understand. One result is the area of the triangle. In most curricula, students are taught that in order to find the area of a triangle, they need to find the altitude and length of the base of the triangle and then divide that product in half. This common method is fast and reliable, but it is not the only one. Heron of Alexandria proved a method of finding the area of a triangle where we can use the perimeter of a triangle in order to find its area. This proof, although significant in its result, tends to be lengthy and difficult to understand initially without a developed mathematical background. As William Dunham noted, "This is a very peculiar result which, at first glance, looks nothing if not a misprint…the formula has no intuitive appeal whatsoever. But it is not just its strangeness that brings it to our attention as a great theorem. Rather, it is the proof that Heron furnished, which is at once extremely circuitous, extremely surprising, and extremely ingenious" (Dunham 1991, pg. 119).

In order to find ways to make this proof more accessible, we analyzed and used Oliver Byrne's methodology presented in his rewriting of Euclid's *Elements*. Byrne's 1847 text, *The First Six Books of the Elements of Euclid in Which Coloured Diagrams and Symbols Are Used Instead of Letters for the Greater Ease of Learners* is the first attempt to illustrate Euclid's *Elements* for the purpose of education. To learn how to relabel and construct diagrams using distinct shapes and colors, we first rewrote the proofs to the infamous Pythagorean Theorem. We then applied this method to develop an original rewritten proof of Heron's Area of a Triangle using Byrne's methodology. Our goal is to demonstrate the beauty of Byrne's approach with this classic result, as well as to make the proof more accessible to a wider range of readers.

2. Review of Relevant Literature and Contexts

2.1 Primary and Secondary Sources

Primary sources provide a rich background into the context of our history, and primary sources within mathematics are no different. They are sources of information that are written without "any interpretation or commentary" such that they "display original thinking, report on new discoveries, or share fresh information" (University of Minnesota 2019). Similarly, secondary sources are works or texts that "offer analysis or restatement of primary sources" whether it is through providing commentary, summarized content, or further explanations (University of Minnesota 2019). Both primary and secondary mathematical sources provide different, but interesting, perspectives for students and learners. Whether it is providing a polished explanation of a result or the original calculations, both types of sources have their benefits.

As a student, these primary sources offer a unique, hands on approach to understanding fundamental mathematical concepts that I don't typically receive when learning through a textbook. Although I can learn the formulas, the results, and how to prove them, I must take that information as it is given to me rather than obtaining that result like previous mathematicians. The beauty of using the original texts is that they allow me to develop my own analysis of the result, in whatever field of mathematics it may be in, and feel more satisfied with the logic and method of that result. Finally, working with primary sources grants me the satisfaction of working through material that can be dense and be able to interpret it in the original way as well as the modern methodology.

2.1.1 Elements

Euclid's *Elements of Geometry* is one of the most infamous and foundational works within mathematics due to the material and methodology presented within the text (Euclid and Fitzpatrick 2005, Pg. 4). It was authored by Euclid of Alexandria in BCE and covers proofs, definitions, and postulates in the fields of geometry, proportion, and number theory across 13 books (Euclid and Fitzpatrick 2005, Pg. 4). Although many of the proofs contained in *Elements* were not authored originally by Euclid, his work in reordering each of these results such that

they build off of five initial postulates and results that have already been proven is what makes Euclid's *Elements* well known as a mathematical text (Euclid and Fitzpatrick 2005, Pg. 4).

2.1.2 Heron's Work

The main result that we are focusing on in this paper is Heron of Alexandria's general formula to calculate the area of a triangle. It offers another perspective on how to calculate a triangle's area as it relies on using the semiperimeter, or the sum of all of the side lengths of the triangle divided by two (Thomas 2011, Pg. 471). This result was written in Heron's text, *Metrica*, in his original words which makes it a primary source (Thomas 2011, Pg. 471).

2.1.3 Byrne's Elements

Oliver Byrne was an Irish mathematician, civil engineer and author who was born in 1810 and passed away in 1880 from bronchial pneumonia (Hawes and Kolpas 2015). In 1847, Byrne published his works on Euclid's *Elements* where he introduced a new approach: replacing all of the labels with colored shapes and lines. These shapes and colors act as the new labels where Byrne, using artistic statements, establishes the relationships between the constructions and proofs within each postulate (Hawes and Kolpas 2015). Byrne's *Elements* is only a rewriting of the first six books of Euclid's *Elements* and therefore is a secondary source.

Byrne published *The First Six Books of the Elements of Euclid in Which Coloured Diagrams and Symbols Are Used Instead of Letters for the Greater Ease of Learner* with one main goal in mind. He notes that his text is meant to "introduce the system rather than to teach ant particular sets of propositions"(Byrne and Rougeux 1847/2018). Given that this approach focuses on artistic choices within a visual proof, he also adds that his methodology is meant to "teach people how to think, not what to think" (Byrne and Rougeux 1847/2018). This all leads to accessibility within math education. With that said, his work primarily focuses on reaching those who are learning geometry or even those who are educating others or themselves within this field (Byrne and Rougeux 1847/2018).

3 Methodology

The goal of this paper is to analyze Byrne's approach and recreate it using another mathematical result. Since Byrne's methodology is very unique to math proofs, we need to understand the decisions and artistic liberties that go into making a visual proof.

Byrne primarily focuses on eliminating words and using almost exclusively distinct shapes and colors to label important results within a proof whereas Euclid uses exclusively labels and written explanations in order to guide the reader through the proof and constructions. Let us consider these differences in **Figure 1** and **Figure 2**.



Let ABCD be a circle, and let ABCD be a quadrilateral within it. I say that the (sum of the) opposite angles is equal to two right-angles.

Let AC and BD have been joined. Therefore, since the three angles of any triangle are equal to two right-angles [Prop. 1.32], the three angles CAB, ABC, and BCA of triangle ABC are thus equal to two right-angles. And CAB (is) equal to BDC. For they are in the same segment BADC [Prop. 3.21]. And ACB (is equal) to ADB. For they are in the same segment ADCB [Prop. 3.21]. Thus, the whole of ADC is equal to BAC and ACB. Let ABC have been added to both. Thus, ABC, BAC, and ACB are equal to ABCand ADC. But, ABC, BAC, and ACB are equal to two right-angles. Thus, ABC and ADC are also equal to two right-angles. Similarly, we can show that angles BADand DCB are also equal to two right-angles.

Thus, for quadrilaterals within circles, the (sum of the) opposite angles is equal to two right-angles. (Which is) the very thing it was required to show.

Figure 1: Screenshot of an excerpt of Euclid's *Elements* on Cyclic Quadrilaterals (Euclid and Fitzpatrick 2005, Pg. 92)

Figure 1 demonstrates the original proposition that Euclid had in his text, *Elements of Geometry*, on Cyclic Quadrilaterals. As we can see, it offers a final construction with labels and then written steps and proof that refer back to the figure. This method is what is normally seen within texts as

it explicitly calls out each component of the construction so the reader can recreate it while also providing a step by step reasoning within the proof. Although this is thorough and can provide the reader steps on how to recreate the result, it is dense and could be more difficult on the reader's understanding of the result.



Figure 2: Screenshot of an excerpt from Byrne's *Elements* on Cyclic Quadrilaterals (Byrne and Rougeux 1847/2018)

Figure 2 demonstrates Byrne's rewriting of Euclid's proposition on Cyclic Quadrilaterals by providing a figure with multiple colors and no labels. From there, he provides a brief description on how to construct the figure and a brief visual proof of the result. This method highlights the important results while also keeping the proof concise and visually appealing. However, it does involve the reader to rely on the final figure for their construction and leaves out important details on why we can say each step of the proof.

Both methods have their benefits and their costs, so we wanted to blend the format of both types of proofs such that we can utilize the visual components of Byrne's proofs while also ensuring there was enough written explanations like in Euclid's proofs so the construction and reasoning can all stand on their own. Fortunately, due to the accessibility within technology today, recreating proofs to blend both of these benefits is significantly easier than in the past. Let us now consider how we were able to create our visual proofs.

3.1 Programs and Process

One of the primary components of a visual proof, especially one that mimics Byrne's proofs, is the figure. Both Byrne and Euclid include a final result of what the construction should look like, but Byrne takes his visualizations a step further by emphasizing the creative freedom and artistic choices within the figures. In order to recreate this type of figure, I decided to use the Notability app in order to draw each of the figures manually due to the functions within the app that can "snap" and "fill" different shapes (Ginger Labs, Inc 2025). Additionally, I used images of the constructions of Heron's Result and traced over those figures in Notability to ensure that the arguments of the constructions could still hold true. Finally, I used a feature in Notability that could convert any handwritten words and explanations into a typed format so the proofs could flow seamlessly within this paper and also be readable for future readers (Ginger Labs, Inc 2025).

For the process of writing and drawing these proofs, there were two distinct ones. For proofs that were already rewritten by Oliver Byrne, I first read and annotated his proof as well as Euclid's proof. This allowed me to familiarize myself with the constructions, arguments, and important results and reasoning behind all of the steps. I also took this as an opportunity to create extra notes on how Byrne and Euclid wrote their proofs so I know what to include or exclude for my own proof. Once I felt comfortable with the proof and the result, I wrote a first draft of the result with the colors and explanations I wanted. After reviewing it and reformatting it, I cleaned up the figures and the written explanations so that it flowed in a way that was accessible but also visually appealing.

When working with Heron's Result, a proof that Byrne did not write himself nor was it contained with Euclid's *Elements*, the process looked slightly different. I still began with reading and annotating Heron's proof and construction, but I used a combination of reading Heron's original proof as well as written commentary on the primary source (Bonsangue & Clinkenbeard 2025). This commentary separates Heron's proof into six distinct steps and expands Heron's words into modern notation and language. This was incredibly helpful in my understanding of the proof since the original text can be dense. Using a similar approach as the one presented in the manuscript, I drafted a visual proof with constructions, important results, and several figures that highlighted important results for every step. Finally, after receiving feedback, I reformatted the proof and finalized all of the figures.

3.2 New Notation

One feature that I noticed was not included in Byrne's proofs was that many of his constructions relied on the final figure rather than stand alone statements. This could be a result of when Byrne published and printed his text since he most likely needed to cut down on statements in order to make it less expensive. That being said, I wanted to add notation so that readers could also recreate the constructions without having the need to rely on the final figure. In order to do this, I decided to follow a similar notation to angles to address how to construct and mark lines and shapes on certain parts of the visualization. Consider **Figure 3** to see an example of new notation that I have developed for the purposes of my proofs.



Figure 3: Example of new notation describing how to construct a line on a square

As shown in **Figure 3**, there are new ways I have described how to mark or construct figures, lines, or angles onto an existing figure within the proof. Byrne does not have this in his original proofs and I felt that this type of notation is beneficial to the reader's understanding of where exactly they would need to mark or construct a line or angle without having the need to rely on my final figure. Ultimately, the new notation that I have used is just to explicitly explain where I want everything to be constructed or marked on my figures.

Additionally, I also used new ways to describe a length or figure. Byrne often uses dashed or solid lines within his work and I wanted to reflect that within my own proofs. With that being said, I decided to mix the dashed and solid lines together to create new "colors" to use. Given the complexity of Heron's proof, I realized that I would need more ways to describe important results while still maintaining consistency throughout the proof. Since I wanted to stay true to using four colors like Byrne did, mixing solid and dashed lines together allowed me to achieve my goal. Consider **Figure 4** to see an example of blending lines.



Figure 4: Example of a "blending" dashed and solid lines within my proof

4. Demonstration

In order to fully understand exactly how to structure visual proofs similarly to Byrne, we first need to practice on a result that is well known. Let us consider the Pythagorean Theorem, a proof and construction found in Euclid's *Elements* and in Byrne's rewritten *Elements*.

The focus of this demonstration was to practice making artistic choices and incorporating them into the construction and proof of a main result. Since there are many decisions regarding the location of colors, the balance of words and shapes, and how to keep the integrity of the proof the same, using a result that Byrne already applied his method to helps us understand exactly what choices we need to make in order to create a visual proof. Additionally, this demonstration also allows us to find a balance between having written explanations of results and purely visual results.

Using the new notation that was established previously, we can create the following visual proof of the construction and reasoning for the Pythagorean Theorem that incorporates reasoning and visual components. Please note that this proof, although similar to Byrne's original proof, focuses on the other "half" of the Pythagorean Theorem proof and offers more explanations.

4.1 Pythagorean Theorem Proof

In a right angled triangle,



We find that our construction is as follows:



We find that our final figure is as follows:



and an equal corresponding angle in between both sides.



5. Heron's Area of the Triangle

Let us now turn our attention to Heron's Area of a Triangle.

5.1 Heron of Alexandria

Heron of Alexandria was a Greek Mathematician whose work became more popular around 62 AD. Generally, he focused on Geometric results and engineering with some of his works including *Metrica*, *Geometrica*, *Stereometria*, and other texts that discuss Geometry (Encyclopedia Britannica). One of his results discussed within *Metrica* was a general formula of finding the area of a triangle that relies on only the side lengths (Encyclopedia Britannica).

5.2 Heron's Result Example

In order to find the area of a triangle, generally students will try to find the height, or the altitude, of the triangle. From there, they would multiply that height to the value of the triangle's base and then divide that result by two. This result is efficient and one that is discussed more often, but there are more. As previously mentioned, Heron of Alexandria found a result and included it in *Metrica* of a general formula to find the area of any triangle (Thomas 2011, Pg. 471). This formula is as follows:

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where *s* is the semiperimeter of the triangle and *a*, *b*, and *c* are the side lengths of the triangle. Note that we can find the semiperimeter of a triangle by taking the sum of all of the side lengths and dividing that sum by two. In other words,

$$s = \frac{a+b+c}{2}.$$

This result ends up giving us the area of the formula and although we need to find that extra step of the perimeter, we do not need to find the altitude of a triangle. Since we may not always be given right angles or values that make the calculations straightforward, this formula can be really beneficial.

5.2.1 Example

Before we go into the proof of Heron's formula, let us first convince ourselves that this result works. For the following calculations, consider **Figure 5**.



Figure 5: Triangle with side lengths a = 6, b = 8, and c = 10

In order to find the area of the triangle in Figure 5, let us first use the original area formula of

$$A = \frac{1}{2}bh.$$

Since the triangle we are finding our calculations from is a right angle, our base and our height is given to us right away. Using the original formula, we get the following calculations:

$$A = \frac{1}{2}(6)(8)$$

$$A = 3(8)$$

$$A = 24.$$

Therefore, we have found that the area of the triangle in **Figure 5** is 24. Now, let us compare this result with the one found in Heron's formula. Consider again Heron's formula, noting that $s = \frac{a+b+c}{2}$ and *a*, *b*, and *c* are side lengths,

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

Let us first find the semiperimeter of the triangle. Using the formula for the semiperimeter, we find that

$$s = \frac{6+8+10}{2}$$
$$s = \frac{24}{2}$$
$$s = 12.$$

Next, using Heron's formula and our new value for *s*, we find the following:

$$A = \sqrt{12(12 - 6)(12 - 8)(12 - 10)}$$

$$A = \sqrt{12(6)(4)(2)}$$
$$A = \sqrt{72(4)(2)}$$
$$A = \sqrt{288(2)}$$
$$A = \sqrt{576}$$
$$A = 24$$

Using Heron's Result, we have now found the area of the triangle in **Figure 5** to be 24, which is the same result that we found when using the original formula. Therefore, both results work and give us the area of a triangle.

Since the example we practiced on was a right triangle, it makes finding the area a bit more straightforward since we can use the original formula due to the height and base being already given to us. This is not always the case and we will have to go through more steps if we were to have a non-right triangle where we do not know the altitude of it initially. Those cases are what makes Heron's Result so desirable since it can work for any triangle as long as we know or can find the side lengths of the triangle.

5.3 Heron's Area of the Triangle Proof



IV.4



are bisections through our construction.



since each triangle side length is shown once.



We now have the following figure. Note that this figure does not have the background triangle markings.



Note that $\bigcirc = \square$.

I. Def. 22

Additionally, we can construct a cyclic quadrilateral and we can use the properties of a cyclic quadrilateral. So,



We now have the following figure. Similarly to step 3, we do not have the background triangle markings here as well.



Now, let us consider the following proportions:



By substitution and rotating the proportions, we find







<u>Step 6</u>

Let us consider our figure once more.



Through a previous result, note that we have found



Next, consider the following



By the relationship outlined above, we find our major result,



Q.E.D.

6. Applications

Although the main result and takeaway from this paper was the rewritten proof of Heron's Result, the application of how we were able to create the visual proof has a variety of different applications.

6.1 Math Education

Math education is incredibly important and finding more engaging or accessible ways to discuss different topics within mathematics allows a broader range of students to experience the benefits that mathematics provides. One barrier that can be seen is how problems are structured and how the explanations of those problems can be intimidating for students. Although the change is small, rewriting angle labels or side lengths in unique, corresponding colors and shapes can help students connect the relationships between different components of the proof.

Additionally, this way of establishing important results and relationships is not exclusive to Geometry or geometric results, although it is more relevant due to the use of constructions and properties within those proofs. Any way of visualizing mathematical relationships, whether that's through shapes, colors, or drawings, can help students understand what is being asked of them and what they need to find.

Finally, introducing visual proofs also provides students with an opportunity to rewrite known proofs and constructions in their own words and colors so that they can not only get a better understanding of the result, but being able to associate and highlight the important reasonings and results from each step (Hawes and Sid 2015). Visual activities for different results can give students an opportunity to have a refreshing activity involving the arts while also allowing them to work on their own reasoning for their proofs and then compare them with their peers. Finally, it can allow students to be intentional with how they want to explain their thoughts and evidence regarding their proofs.

7. Conclusion

Now that we have gone through the proof of Heron's Result, let us consider feedback on this subject, limitations, and where to go from here.

7.1 Initial Feedback and Responses

This research was presented on a poster at the Spring Research Showcase that the University's Undergraduate Research Opportunity Center (UROC) hosted at California State University, Monterey Bay on April 18th, 2025. Since the result was something that was fairly new and different compared to other presentations, I wanted to share the initial feedback and overall response to my presentation.

Given that the overall goals of this project were to work with a primary source and find an accessible way to teach mathematics, my hope was that my audience would be able to follow along and appreciate Heron's Result through the visual proof and still generally understand the steps of each proof. When presenting my work at the Spring Showcase, I had three groups of 5-10 people each where all three groups had different audiences. Whether my audience had members who had advanced technical math knowledge to those who have not worked with mathematics in this lens, the general response was that each group could follow along with the result. Due to the nature of Heron's proof, each step can be dense with its reasoning on the many proportions and relationships within the proof and construction. As Dunhman notes, "Heron displayed...a remarkably rich and elegant proof that boasts one of the best surprise endings in mathematics" (Duhman 1991, pg 119). Overall, those who listened to my presentation were able to use the corresponding shapes and colors to relate the different relationships within the proof in such a way that it gives them a greater understanding and appreciation of the result and enjoy the satisfying ending with me.

7.2 Limitations

Creating visual proofs seem simple initially, but they require a lot of forethought and planning. In Byrne's work, he focused on creating visually appealing figures that also highlight the significant arguments within the constructions and proofs. Therefore, one of the main limitations that we encountered was how we wanted to show off each of the colors and what lengths and values they represented. Given that Heron's result is very complex, we decided that in order to keep the integrity of the proof throughout the visual proof, we could highlight different components of each "step" of the proof and what the main results were for those steps.

Additionally, keeping the visual proof appealing and artistic was another limitation since we wanted to still honor Byrne's methodology in combining the arts and mathematics. There

were many iterations of the final figure for our construction of Heron's Area of the Triangle with different artistic choices to make the figure visually appealing while also maintaining the similar lengths, angles, and properties that are present in the construction.

7.3 Future Work

This timeline of the research within this paper allowed for several disseminations and presentations of the results and methods, but it did not allow for testing the method within a classroom. Although my coordinating professor, Dr. Jennifer Clinkenbeard, and I would use these methods in our explanations and notes, future work that I would find beneficial given the scope of this research would be to walk students through the visual proof of Heron's result and see if students could either follow along or recreate it. Ultimately, using the work that has been completed in analyzing Byrne's method to then recreate it on a new, more complex geometric proof can be applied to other results in the future to make learning mathematics more accessible.

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