François Viète Uses Geometry to

Solve Three Problems

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From Egypt and Greece in ancient times to Iran during the Middle Ages and Italy during the Renaissance, geometry and cubic equations have long fascinated mathematicians all around the world. A cubic equation with the unknown value x has the general form

$$ax^3 + bx^2 + cx + d = 0,$$

where a, b, c, and d are constants. Someone who was also fascinated with geometry and cubic equations was the French mathematician François Viète. Through a deeper look at ancient Greek mathematics coupled with his self-invented "analytic art," Viète developed methods of trisecting an angle, solving a special type of cubic equation, and inscribing a heptagon in a circle.

François Viète was born in 1540 in Fontenay-le-Comte, Poitou (now Vendee), France, and died in Paris, France, in 1603. The son of a lawyer, Viète graduated himself with a law degree from the University of Poitiers in 1560 [1, par 1]. He then worked as a lawyer for four years, during which his "practice appears to have flourished" [4, p. 1], before switching careers to serve as educator and councilor for various members of the French aristocracy and royalty including Antoinette d'Aubeterre, King Charles IX, King Henry III, and King Henry IV during a time of great political and social unrest in France [1, par 2-6]. Overall, Viète was known for his good character, tact, and contemplative, problem-solving mind [2, par 2-5].

Despite never actually being a professional mathematician or scientist by trade, during his time working for the aristocracy and royalty, Viète achieved much in mathematics, including a new algebraic notation system, new revelations in geometry by taking a closer look at ancient Greek geometry, and improvements in the theory of equations [1, par 9-14]. He is sometimes even called the "father of algebra" [1, par 18]. Likewise, among other things, during his time serving Henry IV, Viète used his mathematical abilities to decode messages for the king that were being sent to Phillip II of Spain—the enemy of Henry IV [1, par 8].

Here, we will examine three pieces of Viète's mathematical work from his book

Supplementum Geometria (*A Supplement to Geometry*) [4]. Published in 1593 and contained in

the *Analytic Art*, a larger volume translated by T. Richard Witmer in 1983 that contains other

works on algebra, geometry, and trigonometry by Viète, *A Supplement to Geometry* contains

twenty-five propositions and proofs on geometry. Initially, many of these propositions appear to

just show geometry that is similar to that done by the ancient Greeks. But Viète's genius lies in

his using the geometry coupled with algebra and trigonometry to develop solutions to other

mathematical problems.

Even while developing new algebraic methods, Viète felt it was important to stay true to

the geometry of the ancient Greeks. According to mathematics professors Victor Katz and Karen

Hunger Parshall in *Taming the Unknown: A History of Algebra from Antiquity to the Early*

Twentieth Century [3], throughout his work, Viète "insisted on grounding his algebra

philosophically in geometrical strictures that had persisted since at least the time of Euclid,

geometrical strictures that, viewed through our modern, post-Cartesian eyes, clearly mark Viète

as an algebraist very much rooted in the sixteenth century" [3 p. 240]. This geometry is not

surprisingly present throughout much of Viète's *Supplementum Geometria.*

Three such problems that Viète solves in *Supplementum Geometria* are trisecting an

angle, finding a solution to a special type of cubic equation, and inscribing a heptagon in a circle

[4, pp. 398-415]. The first and the latter problems use and expand on geometric methods that had

already been around since ancient times. Likewise, for the second problem, historically, much

had already been learned in solving various types of cubic equations. Notably, Archimedes, in

ancient Greece, found the solution to a particular cubic equation using geometry and conic

sections, and Omar Al-Khayyami found geometric and algebraic solutions for solving a so-called

depressed cubic in Persia during the middle ages [3, pp. 165-166]. Then in Italy 1539, Cardono found a formula for solving a depressed cubic [3, p. 216]. (By the simple substitution $y = x - \frac{b}{3a}$, the general cubic $ax^3 + bx^2 + cx + d = 0$ is reduced to a cubic equation without the x^2 term, creating a cubic equation called a depressed cubic: $ay^3 + by + c = 0$.) Likewise, during the late 1500's, Viète took a closer look at mathematical works by the ancient Greeks [3, pp. 240], and with that he developed another solution to a particular type of cubic equation, along with solutions to trisecting an angle and inscribing a heptagon in a circle as mentioned.

In Proposition IX, Viète illustrates a method of trisecting an angle. This is the same proposition and proof Archimedes used to trisect an angle using a marked straight edge in ancient Greece. According to Viète, trisecting an angle is a key method for solving cubic equations. In fact, he says that "all cubic… equations, however affected, that are not otherwise solvable can be explained in terms of two problems—one the discovery of two means between given <extremes>, the other the sectioning of [trisecting] a given angle into three equal parts." In Viète's own words, "this is very worth noting" [4, p. 417].

As such, Viète's construction in trisecting an angle in Proposition IX may have suggested to him a method for solving a special type of cubic, which is illustrated in Proposition XVI. The special type of cubic equation that Viète solves is a cubic of the form

$$x^3 - 3x = 1.$$

Further, in Proposition XXIV, Viète walks the reader through two different methods of proving a single proposition—inscribing a regular heptagon (seven sides) in a given circle. Both proofs use geometry to find the solution, and both have interesting features.

Note that all comments in [square brackets] are mine, all comments in <angle brackets> are from the translator, and terms that are not enclosed with parentheses but are listed to a power, such as AB^2, should be interpreted as $(AB)^2$. Further, not all pictures are to scale.
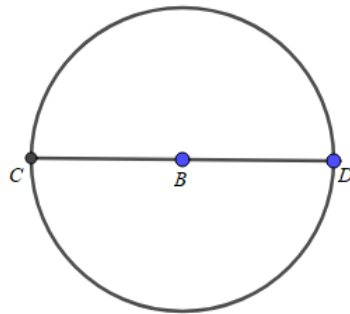

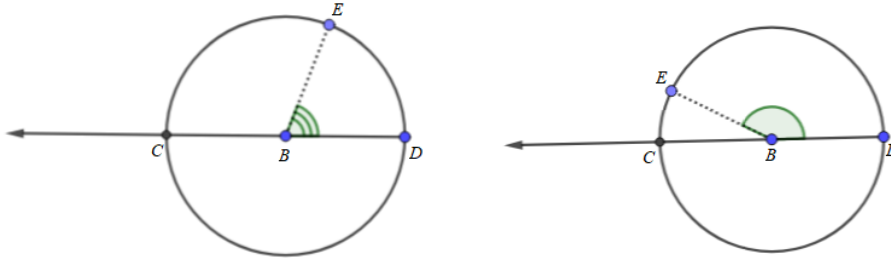Proposition IX [4, p. 398]

To trisect a given angle.
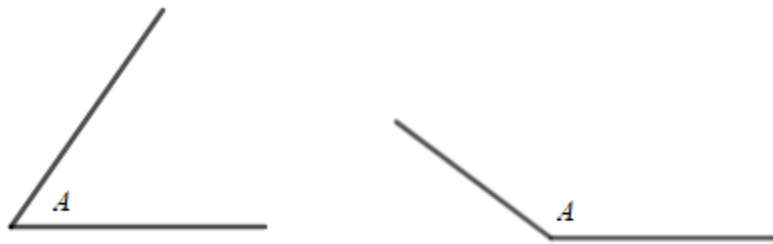
[Construction]

Let A be the angle to be constructed.
From the center B describe a circle at any distance you choose, and let the diameter be CBD.

Mark off the arc DE which defines the size of the given angle [so angle A is congruent to angle EBD] and extend DBC indefinitely.

Draw the straight line *EFG* [in fact, either *EGF* (left sketch) or *FEG* (right sketch)] cutting the extended diameter at *F* and the circumference at *G* so that *FG* is equal to *BC* or *BD*, the radius of the circle [using a straight edge marked with distance *FG* = *BC* = *BD*].
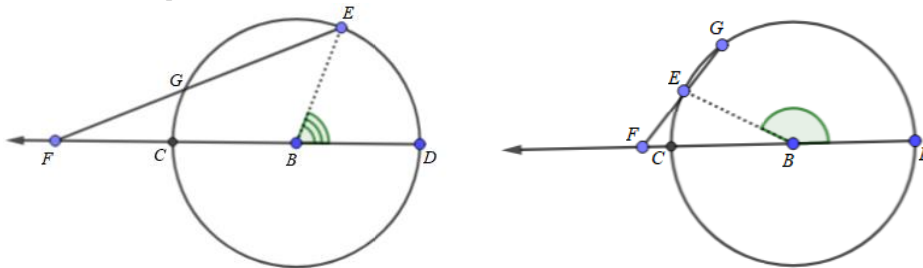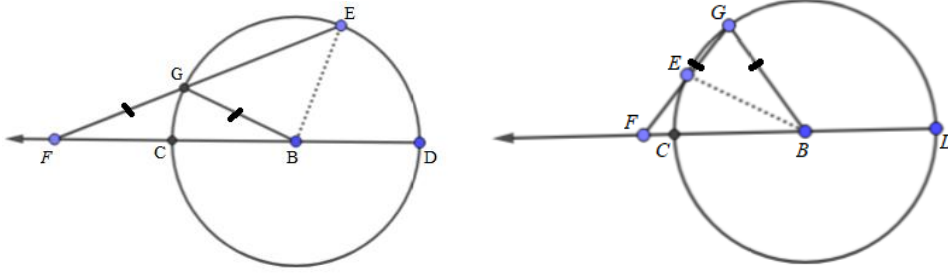

[Claim]

 I say that the angle *EFC* is one-third the angle *EBD*, that is, the given angle *A*, and that the arc *GC* is one-third the amplitude <of the arc *ED*>.
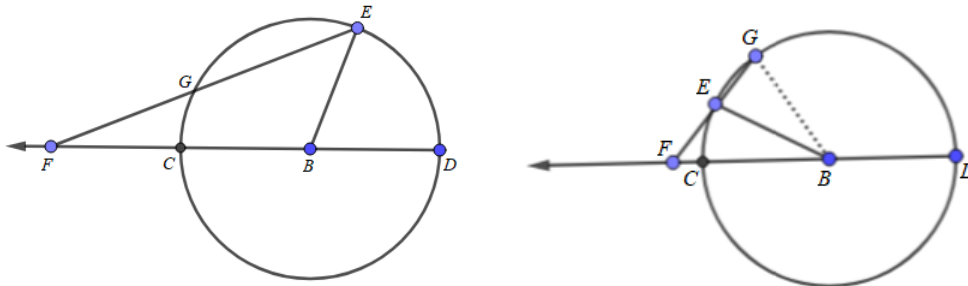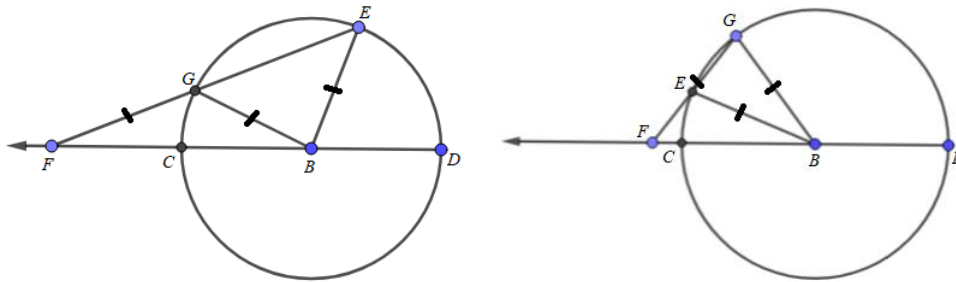

[Proof of Claim]

Let *G* <and> *B* be joined. Then the triangle *FGB* is isosceles [because *GB* = *FG*].
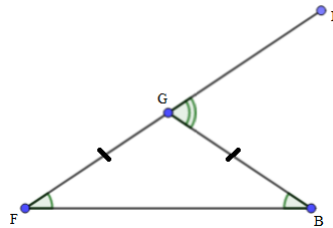
From *B*, one end of its [the triangle *FGB*'s] base, draw *BE* equal [as another radius] to the leg *BG*.

Hence

[The translator calls this a "long leap." Observe, for the acute angle:

(1) By the first part of Euclid I-32, on triangle *BGF*, the exterior angle *BGE* equals the sum of the two interior and opposite angles, *BFG* and *FBE*, namely *BGE* = *BFG* + *FBG* = 2*BFG*.

(2) Because triangle *BGE* is isosceles, we see that 2*BFG* = *BGE* = *BEG*.
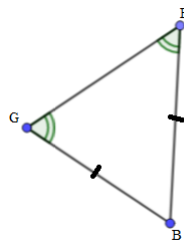

(3) By the second part of Euclid I-32, the sum of the interior angles of a triangle is two right triangles. So, *GBE* = two right angles (in triangle *GBE*) – *BGE* – *BEG*

= two right angles (in triangle *GBE)* – 2*BFG* – 2*BFG*
= two right angles (in triangle *GBE)* – 4*BFG*
= two right angles (in triangle *GBE)* – 4*GBF*.
(4) Also, *GBD* = two right angles (upon semicircle*)* – *GBF*.
(5) Therefore, *EBD* = two right angles – *GBF* – *GBE*

= two right angles – *GBF* – (two right angles - 4*GBF)*

= 3*GBF*.

Observe, for the obtuse angle:

EBD = GFB + *FEB*
= *GFB* + (*FGB* + *GBE)*
= *GFB* + *GEB* + (*GBF* - *FBE)*
= 2*GFB* + *GEB* – *FBE*
= 2*GFB* + *GFB*
= 3*GFB*.
So, *EBD* = 3*GFB*, as desired.]

the angle *EBD* is triple the angle *GBF* or *GFB*.

Moreover, the arc *GC* defines the size of the angle *GBF*. Accordingly, within the arc *DE* mark off the arcs *DH* and *HI* equal to the arc *CG* [using a compass marked with angle *CG*] and draw the straight lies *BH* and *BI*.

Therefore the angle *EBD*—that is, the given <angle> *A*—is trisected by the straight lines *BH* and *BI*, which is what was to be done.

[End of Proof]

As mentioned, the final construction that appears in Proposition IX seems to possibly have suggested the initial construction Viète uses for the following proposition.

<div align="center">

## Proposition XVI [4, pp. 403-405]

</div>

[Theorem]

> If there are two individual isosceles triangles and the legs of one [triangle] are equal [congruent] to those of the other [triangle] and the base angle of the second [triangle] is equal [congruent] to three times the base angle of the first [triangle], [then] the cube of the base of the first [triangle] minus three times the product of the base of the first [triangle] and the square of the common leg is equal to the product of the base of the second [triangle] and the square of the same leg.
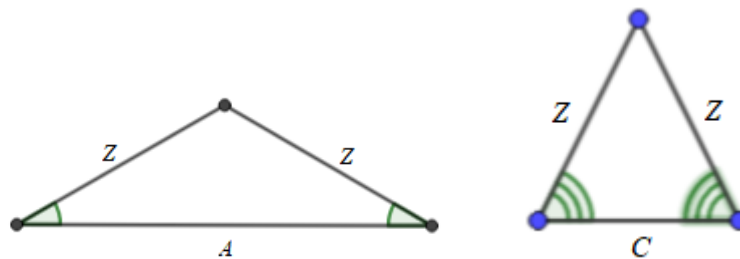
[Claim: $A^3 - 3AZ^2 = CZ^2$]

[Construction]

Let the first [isosceles] triangle be ABC having equal legs AB and BC.
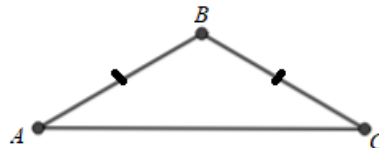
Since the second triangle is also isosceles [by assumption] and either of the [congruent] base angles of this second triangle is three times the angle BAC or BCA and [each of the congruent base angles of the second triangle] is necessarily less than a right angle [By the second part of Euclid I-32, the total angle sum of a

triangle is equal to two right angles.], therefore either of the angles [base] *BAC* and *BCA* [in our first triangle] is less than one-third of a right angle and the angle *ABC* is [necessarily] greater than a right angle.
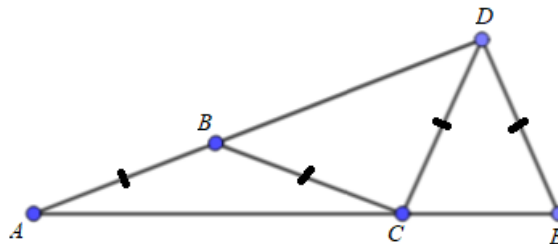
[In the first triangle,

$$ABC + \frac{1}{3} \; right \; angle + \frac{1}{3} \; right \; angle > ABC + BAC + BCA = 2 \; right \; angles.$$
So, $ABC > \frac{4}{3} \; right \; angle > 1 \; right \; angle.$]

Let *AB* and *AC* be extended [to unspecified points *D* and *E*].
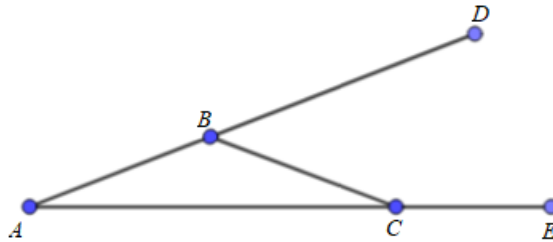
From *C* to *AB* extended, draw *CD* [*D* is now specified] equal to *AB*. Then from *D* to *AC* extended draw *DE* [*E* is now specified] also equal to *AB*. So there are two isosceles triangles, *ABC* and *CDE*.


But *CD* and *DE*, the legs of the second triangle are equal to *AB* and *BC*, the equal legs of the first triangle [all are congruent]. Moreover, just as either of the angles *BAC* and *BCA* is one part of two right angles, so the angle *ABC* <is equal to> two right angles minus those two parts [by the second part of Euclid I-32], and the exterior angle of the angle [triangle] *ABC* <is equal to> those two parts [by the first part of Euclid I-32].
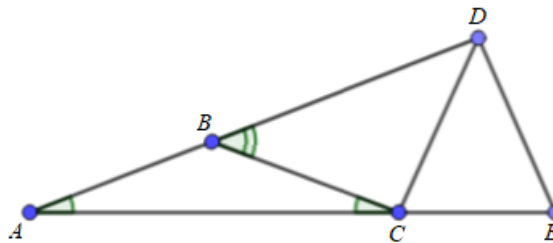
The angle *ADC* is [also] equal to this exterior angle, since the angles *DBC* and *CDB* are equal on account of the equality of the legs *CD* and *CB* [Triangle *BDC* is isosceles. So, by Euclid I-5, "the angles at the base equal one another."]
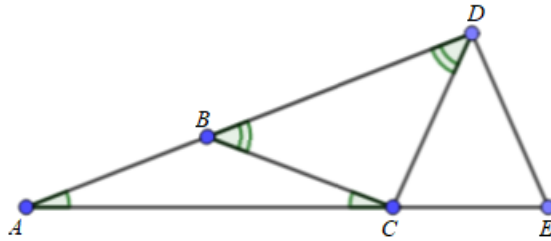
The angle exterior to the angle [triangle] DCA, moreover, is the sum of the angles ADC and DAC [again, by Euclid I-32]. Thus the second triangle [as mentioned in the statement of the proposition] is CDE, which is isosceles and has legs equal to the legs of ABC, the first triangle [as mentioned in the statement of the proposition], and either of its base angles, namely DCE or DEC, is three times the angle BAC or BCA.
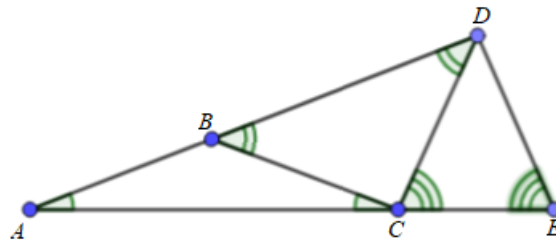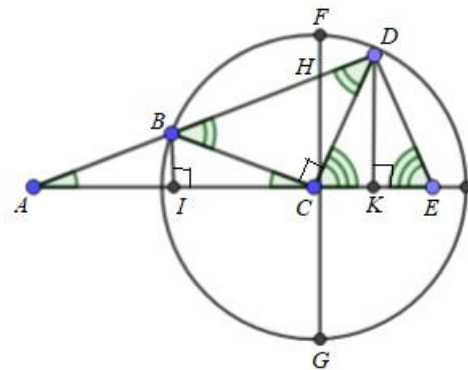

[Proof]

I say, then, that

$$AC^3 - 3(AC \times AB^2) = (CE \times DC^2) \ or \ (CE \times AB^2).$$

[The "or" statement follows since $DC = AB$ by construction.]

[Further Construction]

For let a circle be described at the distance CB or CD from C, its center, and let the diameter FCG cut AE perpendicularly at C and AD [not perpendicularly] at H. Let BI and DK be drawn parallel to FG, cutting AE perpendicularly at I and K.

Hence AI and IC are equal [triangle ABC is isosceles] and AC is twice AI [for later use, $AC = 2(IC)$]. So also AB and BH are equal, making AH twice AB. Likewise CK and KE are equal, making CE twice CK.

[Proof of Claim]

Moreover, CG^2 (that is, AB^2) is equal to $CH^2 + (FH \times HG)$

[Observe:
$$CG = CH + FH$$
So, $$CG^2 = (CH + FH)^2$$
$$= CH^2 + FH^2 + 2(CH)(FH)$$
$$= CH^2 + FH(CH + CH + FH)$$
$$= CH^2 + FH(CH + CG).$$
$$= CH^2 + (FH \times HG).]$$

and, by conversion, $AB^2 - CH^2$ is equal to $FH \times HG$

[Meaning, from above:
$$CG^2 = CH^2 + (FH \times HG)$$
$$AB^2 = CH^2 + (FH \times HG)$$
$$AB^2 - CH^2 = FH \times HG.]$$

(that is, to $BH \times HD$).

[By Euclid III-35, "if in a circle two straight lines [line FG and BD] cut one another, then the [area of the] rectangle [$FH \times HG$] contained by the segments of the one equals [area of the] the rectangle [$BH \times HD$] contained by the segments of the other." So, $AB^2 - CH^2 = BH \times HD$.]
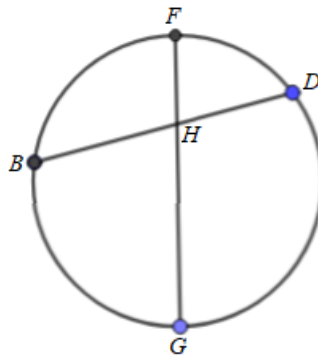
Furthermore, [by the Pythagorean Theorem] CH^2 is equal to $AH^2 - AC^2$ and AH^2 is $4AB^2$ [because $AH = 2AB$].
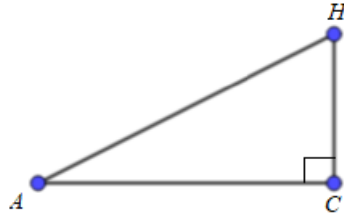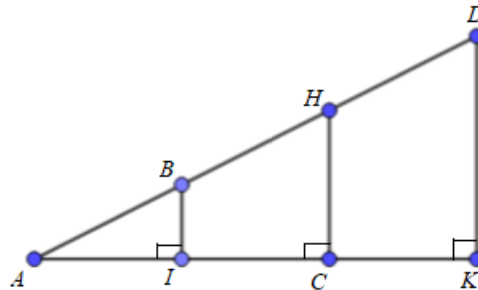
Hence
$$AC^2 - 3AB^2 = BH \times HD.$$

[First by algebra, then using the equalities above, observe:

$$AC^2 - 3AB^2 = AB^2 - (4AB^2 - AC^2)$$
$$= AB^2 - (AH^2 - AC^2)$$
$$= AB^2 - CH^2$$
$$= BH \times HD.]$$

But [by similar triangles]
$$BH : HD = IC : CK$$

$[\frac{BH}{HD} = \frac{IC}{CK}$ from

]

and
$$IC : CK = AC : CE,$$

since the latter terms are twice the former

$$[\frac{IC}{CK} = \frac{2(IC)}{2(CK)} = \frac{AC}{CE}].$$

Hence
$$AC : CE = BH : HD$$

$[\frac{AC}{CE} = \frac{BH}{HD}]$

and consequently AC is to CE as BH^2 (i.e., AB^2) is to $BH \times HD$—i.e., to $AC^2 - 3AB^2$

$$[\frac{AC}{CE} = \frac{BH}{HD} = \frac{BH^2}{BH \times HD} = \frac{AB^2}{BH \times HD} = \frac{AB^2}{AC^2 - 3AB^2}].$$

Thus, resolving this proportion

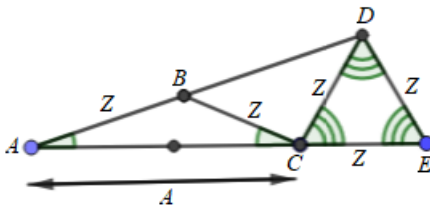[by cross multiplying the equation $\frac{AC}{CE} = \frac{AB^2}{AC^2 - 3AB^2}$],

$$AC^3 - 3(AC \times AB^2) = CE \times AB^2,$$

as was to be demonstrated.

[End of Proof]

[Special Case of the Proposition]

Assuming that Z is any side of an equilateral triangle and that, therefore, each of the angles is one-third of two right angles [$\frac{1}{3} \times 180° = 60°$],

$$A^3 - 3Z^2A = Z^3,$$

[where "A" = AC, "Z" = AB, and "Z" = CE.]
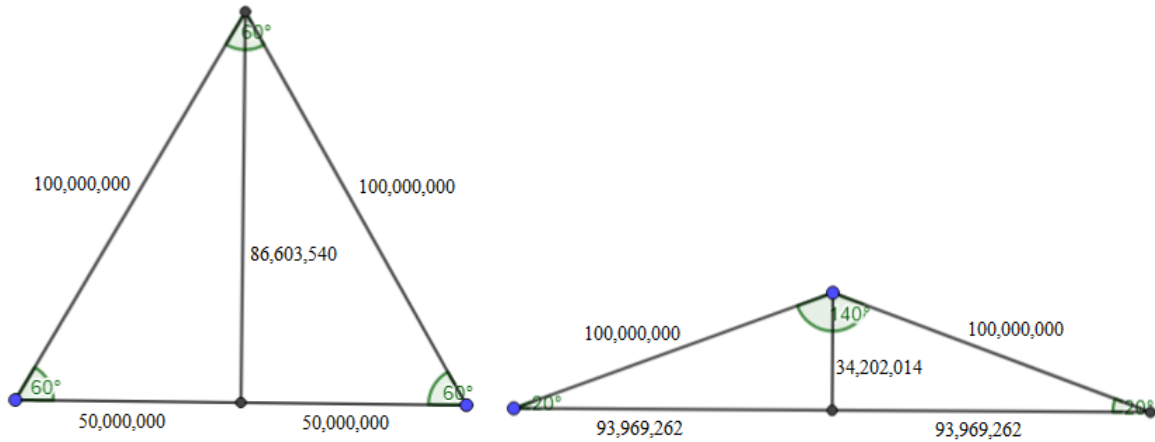
thus making A the base of an isosceles triangle the base angle of which is one-ninth of two right angles [$\frac{1}{9} \times 180° = 20°$],

[Specific Example of That Special Case]

Let Z be 1 and A [be] x. <Then> [$A^3 - 3Z^2A = Z^3$ becomes]

$$x^3 - 3x = 1.$$

If Z is 100,000,000, these are the triangles:

[The equation to be solved here is $A^3 - 3(100,000,000)^2 A = (100,000,000)^3$, with the assumptions that A is the base of an isosceles triangle with base angles of $20°$, and $100,000,000$ is the length of each leg of the triangle. It can be solved for A using trigonometry as follows:

For convenience, let the height of the isosceles triangle be y, and half the base of the isosceles triangle be x. Then, with the assumptions as stated above, the triangle is as follows:
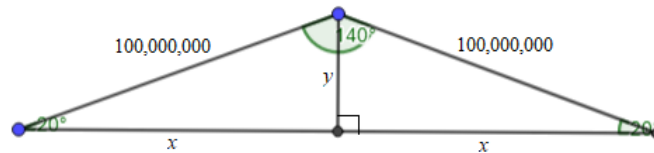
So, $\sin 20° = \dfrac{y}{100,000,000}$.

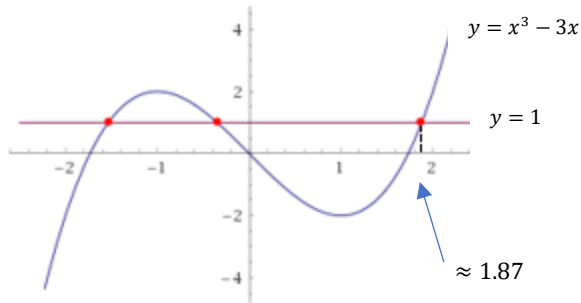$y \approx 34,202,014.$

Likewise, $\cos 20° = \dfrac{x}{100,000,000}$.

So, $x \approx 93,969,262.$

Thus, the base $A = 2 \times 93,969,262,$

meaning $A = 187,938,524.$

So, the approximate positive real decimal solution to

$$x^3 - 3x = 1$$

is $\frac{187{,}938{,}524}{100{,}000{,}000} = 1.87938524.$]

This method can be generalized to solve any cubic equation with the form
$A^3 - 3Z^2A = Z^3$, where A is the base of an isosceles triangle with sides lengths Z,
and the base angles of the triangle are $20°$.]

Just as he did in Proposition XVI, Viète also uses geometry in the following proposition.

Here, Viète illustrates two different methods of proving that a circle (or 360 degrees) can be

divided into seven equal parts. Then he uses this fact to show that any angle can be divided into

seven equal parts.

Proposition XXIV [4, pp. 413-415]

To inscribe an equilateral and equiangular heptagon [seven-
sided regular polygon] in a given circle.

[Construction]

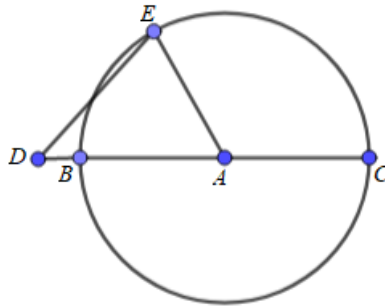Let the given circle have A as its center and BAC as its diameter.
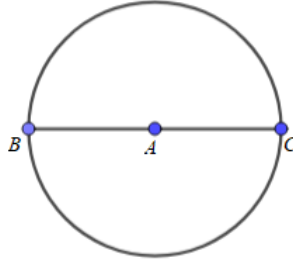
An equilateral and equiangular heptagon is to be inscribed in it.
Extend the diameter, *CB*, to *D* so that

$$DB : DA = AB^2 : DC^2$$
$$[\frac{DB}{DA} = \frac{AB^2}{DC^2}.$$
This is possible by Proposition XIX.]

and draw *DE* across the circumference equal to the radius [*DE* = *AB*].
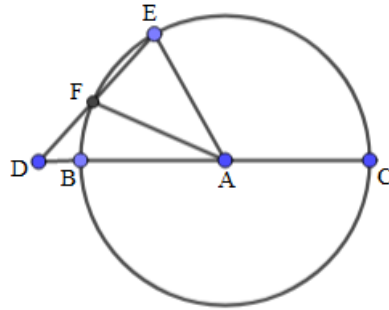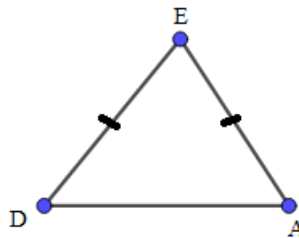

[Claim]

I say that [arc] *EB* is the arc of a heptagon, that is, one-seventh of the whole circumference.

[Proof # 1]

Let *DE* cut the circle at *F* and connect the radii *AE* and *AF* [so, *DE* = *AB* = *AE* = *AF*].

The triangle *DEA*, therefore, is isosceles

and is so constituted that the difference between the base and the leg
[*DA – DE*] is to the base [*DA*] as the square of the leg [*DE²*] is to the
square of the sum of the leg and the base [(*DE* + *DA*)²].

$$[\text{Since } \frac{DB}{DA} = \frac{AB^2}{DC^2}, \text{ and } AB = DE = AC,$$

$$\frac{DA - DE}{DA} = \frac{DB}{DA} = \frac{AB^2}{DC^2} = \frac{DE^2}{(AC + DA)^2} = \frac{DE^2}{(DE + DA)^2} \cdot]$$

Hence the straight line [segment] *AF*, which [as a radius] is equal to the
leg, bisects [by Proposition XXI] the base angle [*A*] and, therefore, just as
its two right angles [of diameter *BAC*] have [can be divided into] seven [equal]
parts, so the angle *EAD* [having been bisected by *AF*] has two [of those seven
parts]

[By Proposition XXII, since in triangle *EAF*,

angle *AEF* = angle *AFE* = 3 × (angle *EAF*),

then the angle sum of triangle *EAF* is 180° = 7 × (angle *EAF*).

So, angle *EAF* = $\frac{1}{7}$ × 180°. Hence, angle *EAD* = 2 (angle *EAF*) = $\frac{2}{7}$ × 180°.]

and, as the four right angles—that is, the whole circumference—have
[can be likewise divided into] seven [equal] parts [each of which is double a
seventh part of angle *EAD*], so the angle *EAD* has one [of those double-sized

seven parts]. But the amplitude of this angle, *EAD*, defines the arc *EB*. It [therefore] subtends one-seventh <of the whole>.

So the arc *EB* is one-seventh of the whole circumference and seven times this subtends <the whole>. Hence there has been inscribed in the given circle an equilateral and equiangular [regular] heptagon, which is what was to be done.
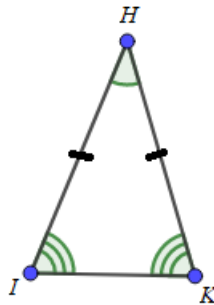
[End of Proof # 1]

Alternatively

> To inscribe an equilateral and equiangular [regular] heptagon in a given circle.

[Construction]

Let the given circle be *ABCDEFG* [labels to be specified later]. An equilateral and equiangular heptagon is to be inscribed in *ABCDEFG*.
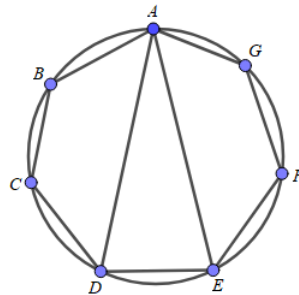
[Proof # 2]

Construct an isosceles triangle, *HIK*, having the angles at *I* and *K* three times the remaining angle at *H*.

Inscribe in the circle *ABCDEFGH* [A = H] a [similar] triangle with the same angles as *HIK* [By Euclid IV-5, "about a given triangle to circumscribe a circle"]. Let this be *ADE* and such that the angle *DAE* is equal to the angle at *H* and that *ADE* and *AED* are equal to those at *I* and *K*.
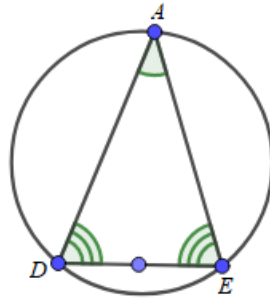
Either *ADE* or *AED*, therefore, is triple the angle *DAE* [by construction]. Hence either the arc *AD* or the arc *AE* will be triple the arc *DE* and one-third of the arc *AD* or *AE* will be equal to the arc *DE*. Let these thirds be *AB*, *BC*, *CD*, *AG*, *GF*, and *FE* and <draw the chords that> subtend <them>.
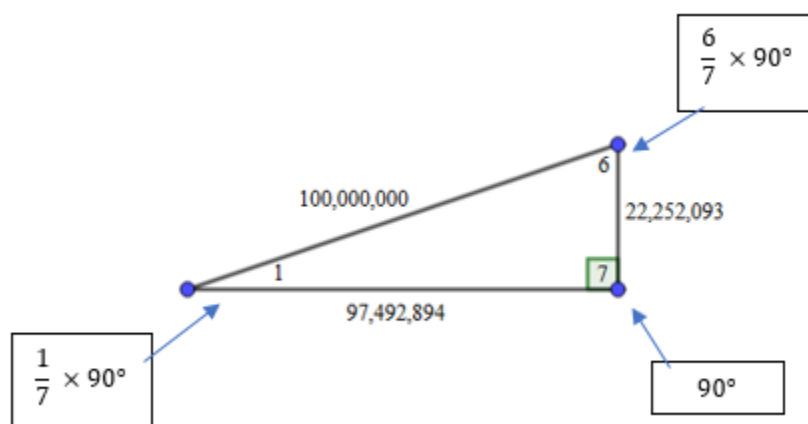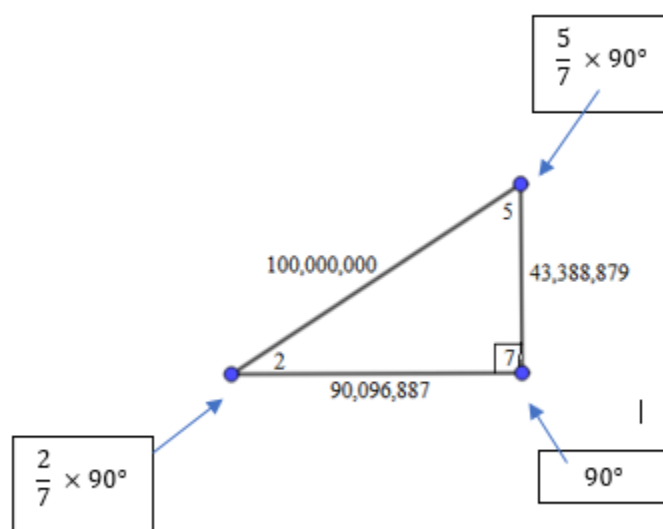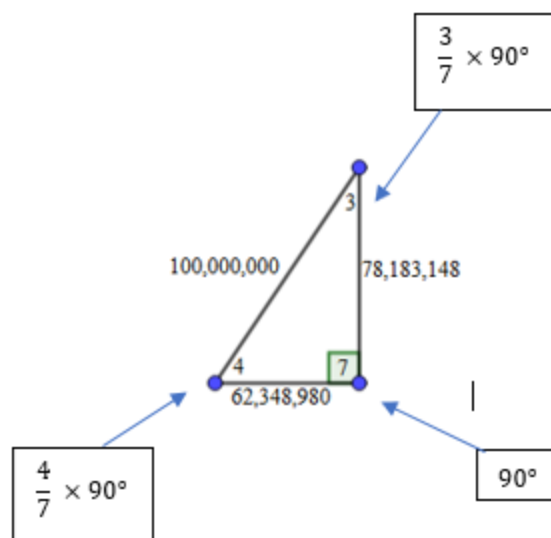

Hence, as was required, an equilateral and equiangular [regular] heptagon has been inscribed in a given circle.

[End of Proof # 2]

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So, it seems that following Proposition XXIV allows one to divide any circle into seven equal parts. Hence, any angle can be divided into seven equal parts, which is exactly what Viète does in the following example.

___

[Example Using Proposition XXIV]

Assume a hypotenuse of 100,000,000 and a right angle with [divided into] seven parts [$\frac{90}{7}$ degrees]. The [three possible] right triangles having seven <parts> will be these:

$\frac{3}{7} \times 90°$

3

100,000,000   78,183,148

4   7
62,348,980

$\frac{4}{7} \times 90°$

90°

$\frac{5}{7} \times 90°$

5

100,000,000   43,388,879

2   7
90,096,887

$\frac{2}{7} \times 90°$

90°

$\frac{6}{7} \times 90°$

6

100,000,000   22,252,093

1   7
97,492,894

$\frac{1}{7} \times 90°$

90°

Using the three right triangles above, Viète then goes on to develop and solve another cubic equation, which is not illustrated here.

In conclusion, François Viète had a fascinating life, and he created a lot of mathematics. His efforts helped not only to expand the mathematical ideas and methodologies of his day but also to pave the way for further advancements in solving algebraic equations and improving analytic geometry. In sum, Viète was an exceptional mathematician with an exceptional mind, and an exceptional body of work.

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