Overcoming Obstacles:

The Lives of Sophie Germain and Sonya Kovalevskaya

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For centuries, strides have been made towards equal educational rights for women worldwide. While the finish line has not yet been reached in some parts of the world, the overall situation for women, particularly in higher-level education, has improved significantly. Today, many people argue that the gender inequality problem still exists in science and mathematics. While women are still underrepresented in mathematics, the reason for this gender gap does not necessarily lie within the mathematical community. Rather, one might claim that the issue stems from the isolation of the mathematical community from the rest of the world.

While other STEM disciplines receive ample media coverage, mathematics lacks this outlet because research in mathematics does not require the same level of public funding or have as immediate of an effect on society [7]. Therefore, the general public's knowledge of mathematics is largely informed by the subjects covered in high school core math requirements. Because of stigmas against women in academia during the foundational eras of mathematics, men discovered the majority of what is learned in grade school. Moreover, many modern advances in mathematics are considered far beyond the scope of a high school classroom, so students are less likely to know about current mathematical research and women's roles in it. In this way, societal attitudes towards mathematics in general have hindered widespread knowledge of women's accomplishments in mathematics.

Because mathematicians as a whole are underrepresented in the media, female mathematicians are even less visible in popular culture. This may lead society to construct a gendered view of mathematics as a subject, informed by poor portrayals of diversity within the mathematical community. This creates a cycle with less women entering and remaining in mathematics and, therefore, fewer female mathematicians. Historically, women's greatest obstacles to becoming mathematicians have been societal bans on women's access to upper level education, societal views on highly educated women, and the influence of these societal views on academia. After overcoming these obstacles, several women have had the privilege of encountering contemporaries who saw their mathematical potential and encouraged them in their studies. Many of these women have been recognized by the mathematical community at large for their groundbreaking work. Not all women had these opportunities, but two women who fought to gain access to upper level education and, ultimately, triumphed over the prejudices against them were Sophie Germain and Sonya Kovalevskaya. Both gained access to higher education despite the societal obstacles, which sought to keep them out, and ended up winning over much of the mathematical community with their work, collaborating with some of their most well-known mathematical contemporaries.

Sophie Germain was born in Paris, France on April 1, 1776 to Ambroise-François Germain and Marie-Madeline Germain née Gruguelin[6]. She had two sisters, Marie-Madeline and Angélique-Ambroise. Her father was a silk merchant and a representative in the États-Généraux of 1789, a group made up of nobility, clergy, and commoners tasked with making decisions on new taxes for France [1]. This committee was necessary to deciding how to pay off the expenses incurred during the wars of the previous century. As the meetings progressed, the representatives from the Third Estate were not satisfied with how decisions were being made and rising tensions amongst the classes eventually lead to the Revolution of 1789 [4]. Four years later, the period of the French Revolution known as the Reign of Terror began [15].

Because of the dangerous conditions outside, Germain was generally kept in the house, and she spent much time in her father’s library. In her readings, she came across the story of Archimedes’ death [12]. According to the most common version of the story, Archimedes was so absorbed in a geometry problem that he did not realize that his city had been overtaken by

Roman troops. When Archimedes, distracted by his work, told an approaching soldier not to disturb his diagram, the insulted soldier killed him. Awestruck at the mathematician's dedication, Germain decided to devote herself to the study of mathematics. Her determination was tested by her parents who worried that too much study would be harmful to their daughter.

While it was in vogue at this time for upper-class women to engage in intellectual conversation, Germain's parents held to the belief that such pursuits would place to much of a strain on their young daughter's mind. With this concern as their motivation, they took away her fire, light, and clothing hoping to deter her from her studies. When they found her one morning asleep at her desk, wrapped in her bedclothes with a frozen inkwell, Germain's parents realized that there would be no stopping their daughter from pursuing mathematics and relented. Germain was allowed to continue her studies openly and absorbed as much as she could of the mathematics being developed around her.

In 1794, L'Ecole Polytechnique was founded to educate the next generation of French scientists and mathematicians, but women were not allowed to attend lectures there. Germain, determined to gain the best education possible, took on the pseudonym Monsieur LeBlanc. LeBlanc had been a student at the school but left and neglected to tell the administration [18]. Germain saw her chance and used M. LeBlanc's name to access notes from lectures, including those of Joseph Louis Lagrange. In addition, she used her pseudonym to submit some of her observations to Professor Lagrange. This was a common practice amongst students at L'Ecole Polytechnique, and upon reading Germain's submissions, Lagrange requested a meeting with the student who had produced them. Germain was apprehensive in revealing her gender due to negative social views on women in academia. Contrary to Germain's fears, Lagrange respected her as a budding mathematician regardless of her gender and continued to offer his support in her pursuit of mathematics. As she continued her studies, Germain went on to exchange correspondences with great mathematical thinkers of her time including Carl Friedrich Gauss.

In her letters to Gauss, Germain resumed the use of her pseudonym, M. LeBlanc. Despite her acceptance by other male contemporaries, she still feared that they would look down on her because of her gender. One of her letters to Gauss expressed her "temerity in troubling a man of genius when [she had] no other claim to his attention than an admiration necessarily shared by all his readers" [18]. In their letters, Germain and Gauss discussed work that Germain had done in the field of number theory. During the early years of their correspondence, Napoleon Bonaparte rose to power in France and set a course to build an empire [10]. Among the countries embroiled in the Napoleonic Wars was Gauss's home, Prussia. While still keeping up the charade of being a male Polytechnique student, Germain discovered that Gauss was living in Hanover, a French occupied town [12]. Fearing for her colleague's life after remembering Archimedes sudden death, Germain asked a family friend to make sure that the mathematician was all right. Gauss thanked the young woman for her concern though he had no knowledge of who she was because he was still unaware that M. LeBlanc was actually Sophie Germain. After this episode, Germain wrote to Gauss explaining who she was and giving her reasons for lying about her gender, and like Lagrange, Gauss continued to support her, impressed by the young student's work. In fact, in a letter to Germain, Gauss wrote of his shock at learning that she was a woman and recognized that "when a person of the sex which, according to [the] prejudices and customs [of the time], must encounter infinitely more difficulties than men to familiarize herself with these thorny researches, succeeds nevertheless in surmounting these obstacles… she must have the noblest courage, quite extraordinary talents and superior genius" [18].

Germain's most significant work in number theory included her research on Fermat's Last Theorem, a problem that had "often tormented her" [16].

Fermat's Theorem: $x^n + y^n = z^n$ has no positive integer solutions for x,y z when $n > 2$.

Her early efforts included an attempt to prove that for all odd prime exponents p, there exists an infinite number of auxiliary primes with the form $2np + 1$ such that the set of non-zero p-th power residues of $x^p mod(2np + 1)$ does not contain any consecutive integers [16]. **Power residues** correspond to the value a in the congruence $x^n \equiv a\ mod(m)$. If the congruence is solvable, then a is a residue of degree n modulo m [14].

Applying this to Germain's problem, consider a case where $n = 2$ and $p = 7$, then $p = 2(2)(7) + 1 = 29$ which is an odd auxiliary prime of 7.

$\{ 17, 27, 37, 47, 57, 67, 77, 87, 97, 107, 117, 127, 137, 147, 157, 167, 177, 187, 197, 207, 217, 227, 237, 247, 257, 267, 277, 287\}\ mod\ 29$

$\equiv \{1, 128, 2187, 16384, 78125, 279936, 823543, 2097152, 4782969, 10000000, 19487171, 35831808, 62748517, 105413504, 170859375, 268435456, 410338673, 612220032, 893871739, 1280000000, 1801088541, 2494357888, 3404825447, 4586471424, 6103515625, 8031810176, 10460353203, 13492928512\}\ mod\ 29$

$\equiv \{1, 12, 12, 28, 28, 28, 1, 17, 28, 17, 12, 17, 28, 12, 17, 1, 12, 17, 12, 1, 12, 28, 1, 1, 1, 17, 17, 28\}$ $mod\ 29$

$\equiv \{1, 12, 17, 28\}\ mod\ 29$

Therefore, the non-zero 7th power residues of $x^7\ mod\ (29)$ are 1, 12, 17, and 28. As nonconsecutive integers, these values satisfy Germain's condition. Continuing in this way, one can find the qualifying auxiliary primes for each odd prime number. It was later proved that there are only a finite number of auxiliary primes that satisfy the nonconsecutive p-th power residue condition, so Germain's solution as stated above could never succeed.

After several failed attempts, Germain offered a proof for prime numbers $2 < n < 100$ for Case 1 of Fermat's theorem. This case states that for the equation $x^n + y^n + z^n = 0$ no integer values for x,y, and z exist such that these values are not divisible by n. The constraints that provide the basis for her proof have become known as Germain's Theorem [6].

Germain's Theorem: Let n be an odd prime. If there is a secondary ("auxiliary") prime p with the properties that

(1) $x^n + y^n + z^n \equiv 0\ (mod\ p)$ implies that $x \equiv 0$ or

$y \equiv 0$ or $z \equiv 0\ (mod\ p)$, and

(2) $x^n \equiv n\ (mod\ p)$ is impossible,

then Case 1 of Fermat's Last Theorem is true for n.

Germain's Theorem can be proved for auxiliary primes $p = 2n + 1$ using Fermat's Little Theorem [3].

Fermat's Little Theorem: Let p be a prime that is a positive integer that if relatively prime to p. Then,

$$a^{p-1} \equiv 1\ mod\ p.$$

Now, consider a prime n with an auxiliary prime $p = 2n + 1$.

$$a^{2n+1} = a^p$$

which can be multiply by a^{-1}on each side to produce

$$a^{2n} = a^{p-1}.$$

Taking into account Fermat's Little Theorem,

$$a^{2n} = a^{p-1} \equiv 1\ mod\ p$$

which simplifies to

$$(a^n)^2 = a^{p-1} \equiv 1 \; mod \; p.$$

Therefore,

$$(a^n)^2 - 1 \equiv 0 \; mod \; p$$

which factors into

$$(a^n + 1)(a^n - 1) \equiv 0 \; mod \; p.$$

Because p is a prime, $a^n \equiv 1 \; mod \; p$ or $a^n \equiv -1 \; mod \; p$, meaning that x, y, or z must be congruent to $0 \; mod \; p$ or

$$x^n + y^n + z^n = \pm 1 \pm 1 \pm 1.$$

Therefore conditions (1) and (2) are proven for Germain's Theorem because $x^n + y^n + z^n$ will never be congruent to $0 \; mod \; p$ unless x,y, or z is congruent to $0 \; mod \; p$ and x^n will never be congruent to $n \; mod \; p$. Through her earlier work, Germain had also proved the cases where the auxiliary prime p took the forms $4n + 1, 8n + 1, 10n + 1, 14n + 1$, or $16n + 1$. Adrien-Marie Legendre later confirmed that conditions (1) and (2) of Germain's Theorem hold for these cases.

Besides her theorem, the concept of Germain primes also came out of her work proving Fermat's Last Theorem. Germain primes are those primes p for which $2p + 1$ is also a prime. Some examples of Germain primes are 2, 3, 5, 11, 23, etc. Over the years, mathematicians have worked to find larger and larger Germain primes. The most recent discovery of one was in March 2010 with 79911 digits. In addition to Germain primes, there are numbers known as sophiens [6]. A sophien was defined by E. Dubouis to be a prime $p = kn + 1$ such that n is also a prime that satisfies $x^n \equiv y^n + 1$ (mod p). These numbers were named in honor of Germain's work.

In addition to her work in number theory, Germain contributed to work on elasticity [12]. In 1811, the French Academy offered a prize for a mathematical explanation of the observations of Ernest Chladni. Chladni took note of the patterns left when vibrations were sent through a piece of metal covered in a layer of sand but did not have the mathematical background to understand what was happening. The 1811 competition received only one entry, which was submitted by Sophie Germain. While her theory was headed in the right direction, Germain's work was incorrect. The Academy offered two more competitions on the same subject in 1813 and 1816 where Germain received an honorable mention and, finally, the prize in spite of continued concerns about the rigor of her work. While the support of her contemporaries had aided Germain in her learning and success, it lacked "substantive criticism from which she might learn" [6]. Many recognized her genius, but the rigor of her work was not necessarily on par with the mathematicians around her. Regardless, her work, including that which pertains to elasticity, has been used in further developing mathematics research for the past several decades.

While Sophie Germain's contributions were significant to the field of elasticity, she did not receive the credit she was due. The most egregious example of this took place when the Eiffel Tower was built. One of the most important components in constructing the famous French landmark was an understanding the elasticity of metals. However, when recognition was given to those that had contributed by inscribing their names on the base of the tower, Germain's name was nowhere to be found. H. J. Mozans commented on this 24 years after the opening of the Eiffel Tower: "Was [Germain] excluded from this list for the same reason that Agnesi was ineligible to membership in the French Academy – because she was a woman? It would seem so. If such, indeed, was the case, more is the shame for those who were responsible for such ingratitude toward one who had deserved so well of science, and who by her achievements had won an enviable place in the hall of fame" [18].

Along with her work in mathematics, Germain had also developed an interest in philosophy and continued her work in both until her death in 1831. Doctors diagnosed Germain with breast cancer in 1829, and despite her suffering and the battles of France's July Revolution being fought around her, she continued to work for the last two years of her life and passed away on June 27th, 1831 at the age of 55. She died before ever having the chance to meet her long-term mentor Gauss. However, he made sure that Germain was honored with an honorary degree from the University of Gottingen. Forty-four years later, the same institution granted a PhD in absentia to Sonya Kovalevskaya.

Sonya Kovalevskaya was born January 15, 1850 in Moscow Russia to Vasilii Korvin-Krukovskii and Elizaveta Shubert [8]. As the middle child stuck between her older sister, Aniuta, and younger brother, Fedya, she suffered as many do from a sense of inadequacy. However, from a young age, she showed promise in the area of mathematics. Stories of her initial encounters with mathematics tell of her Uncle Peter who spoke to her about mathematics, and multiple sources recount the wallpapering of her childhood room with lithographed notes from an integral calculus course by Ostrogradsky [2]. Unlike Sophie Germain's parents, Kovalevskaya's were supportive of their daughter's curiosity where mathematics was concerned.

As she grew up, Kovalevskaya, along with her sister, became interested in political, social, and cultural radicalism. In particular, the movement that Sonya and Aniuta supported was based on the burgeoning philosophy of nihilism and rejected all forms of authority including that of the church, state, and family. Nihilist philosophy promoted equality for women and breaking free of one's parents' grasp, especially in order to pursue an education and career that could have an impact on society. Pursuing equality and a better education is a noble goal, and yet the negative reputation that an association with nihilism could cast upon an individual outweighed these positives.

However, the method by which women were advised to achieve freedom from their parents was alluring to Kovalevskaya who had a desire to pursue a university education but lived in Russia which, like France, excluded women from studying at universities. This situation was further complicated by Russian rules that young women could not travel alone without the written consent of their fathers or husbands, and most fathers would not agree to let their daughters go abroad in order to receive a higher education. This was also true of Sonya's parents. Generally, these young women would gain their independence by legally marrying a man who was also a nihilist and having him give consent for them to go study in another country. Following this pattern, Sonya tricked her father into letting her marry Vladimir Kovalevskaya despite her older sister being unmarried [11]. After six months living in St. Petersburg, the couple moved to Heidelberg, Germany and Kovalevsky's sister, Aniuta, and friend, Iulia Lermontova, soon joined them.

While studying in Heidelberg, Kovalevskaya made connections with famous scientists and mathematicians such as Hermann von Helmholtz, Paul DuBois-Raymond, and R. Wilhelm Bunsen. This was necessary in order for her to be able to study because the school required her to get permission from each professor to attend their class, and she did so, earning the respect of her professors [8]. Furthermore, the connections made through this admissions process, in addition to others that she would make as her education progressed, led to her being a key mediator in later interactions within the European mathematical community. While she did well during her time in Heidelberg, Kovalevskaya only spent three semesters there before moving to Berlin for the next three years to study under Weierstrass with whom she would continue to collaborate for the rest of her life. Originally, Weierstrass was hesitant to take her on as a student despite glowing recommendations from her former professors. The university at Berlin did not accept female students, and frankly, he doubted her mathematical abilities. However, after she impressed him with her work on a problem set which he had given her as a test, Weierstrass accepted Kovalevskaya as a private student [11].

By the spring of 1874, Kovalevskaya had written three doctoral dissertations on a proof of the Cauchy-Kovalevskaya Theorem, the reduction of abelian integrals to elliptic integrals, and the shape of Saturn's rings. Her professors, recognizing her talent, helped her to petition Gottingen University for a doctorate in mathematics, which she received *in absentia* in the spring of 1874. In the fall of that year, Sonya and Vladimir went home to Russia with the goal of teaching at a university there. However, they had difficulty due to the nature of nihilist politics, which at this point often promoted terrorism and assassination. At this time, they consummated their marriage, which had been platonic up to that point, and conceived a daughter whom they nicknamed Fufa. Kovalevskaya took the next several years off in order to raise her daughter and pursue her love of creative writing.

After her five-year hiatus, Kovalevskaya got back in touch with Weierstrass and worked on reviving her mathematical career. This return to mathematics kept her busy as her marriage deteriorated. While she seemed to be climbing back up the ladder of academia, Vladimir's poor financial decisions were catching up to him, and he eventually committed suicide three years after their separation in 1880. Meanwhile, Mittag-Leffler had been fighting the administration at the university in Stockholm to hire Kovalevskaya, and by 1884, she began lecturing in Sweden. Within the year, she was promoted to a position equivalent to that of an assistant professor.

In 1888, her work on the rotation of a rigid body around a fixed point was recognized by the French Academy of Sciences with the Prix Bordin. Known today as Kovalevskaya's top, this is the third and most difficult problem of its kind. Euler and Lagrange had already solved the first two. Euler's problem considered a body with a center of mass not at the center of the body but within the rotational axis, and Lagrange's focused on a body rotating about its center of mass which was both at the center of the body and on the axis of rotation. Kovalevskaya's top had a center of mass not located at the center of the body or on the axis of rotation.

The top which Kovalevskaya considered has principal moments of inertia (I_1, I_2, I_3) such that $I_1 = I_2 = 2I_3$ and $I_3 = 1$, center of mass (x_0, y_0, z_0) such that $y_0 = z_0 = 0$ [9]. Kovalevskaya solved this problem using Newton's Second Law which states that "the acceleration of an object is directly proportional to the net force acting on it and inversely proportional to its mass" [17]. However, another method exists for solving more complex cases like Kovalevskaya's top [13]. Instead of applying Newton's Second Law, one can use Hamilton's Variational Principle:

> **Hamilton's Variational Principle:** The motion of the system from time t_1 to time t_2 is such that the line integral (called the action or the action integral),
>
> $$I = \int_{t_1}^{t_2} L\, dt$$
>
> where $L = T - V$, has a stationary value for the actual path of the motion.

Here $\boldsymbol{L}$ represents the Lagrangian while $\boldsymbol{T}$ and $\boldsymbol{V}$ are the object's kinetic and potential energies respectively. Therefore, L is a function of the object's position $\boldsymbol{q}$, velocity $\boldsymbol{\frac{dq}{dt}}$ or $\dot{\boldsymbol{q}}$, and time $\boldsymbol{t}$. Because the value for the path of motion remains stationary between t_1 and t_2, $dI = 0$. This requires that the net force for the scenario be conservative. With these assumptions made, one can derive the equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$

The movement of a Kovalevskaya top can be described in terms of a change in the Eulerian angles between a stationary system and the rotational system of the top, also known as the body system. Euler angles describe three angles of rotation [5]. One rotation about the z- axis gives a new set of intermediate axes $\xi, \eta,$ and ζ that correspond to the $x, y,$ and z axes respectively. The second rotation takes place about the ξ axis producing a second set of intermediate axes $\xi', \eta',$ and ζ' . As the intersection between the xy and $\xi'\eta'$ planes, ξ' is referred to as the line of nodes. Finally, the system is rotated about the ζ' axis to produce a coordinate system with axes $x', y',$ and z'. The angles formed by each of these rotations are $\boldsymbol{\phi}, \boldsymbol{\theta},$ and $\boldsymbol{\psi}$ respectively. $\dot{\boldsymbol{\phi}}$ is the processional velocity of the top as the center of mass moves around the stationary axis of rotation z, $\dot{\boldsymbol{\theta}}$ is the tipping velocity of the top as it bobs up and down, and $\dot{\boldsymbol{\psi}}$ is the rotational velocity of the top about its axis of rotation z' [13].

Each of these rotations can be expressed as matrices **D**, **C**, and **B** respectively where

$$D = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, the changes in position can be expressed by

$$\xi = \boldsymbol{D}x,$$

$$\xi^{'} = \boldsymbol{C}\xi,$$

$$x^{'} = \boldsymbol{B}\xi^{'}.$$

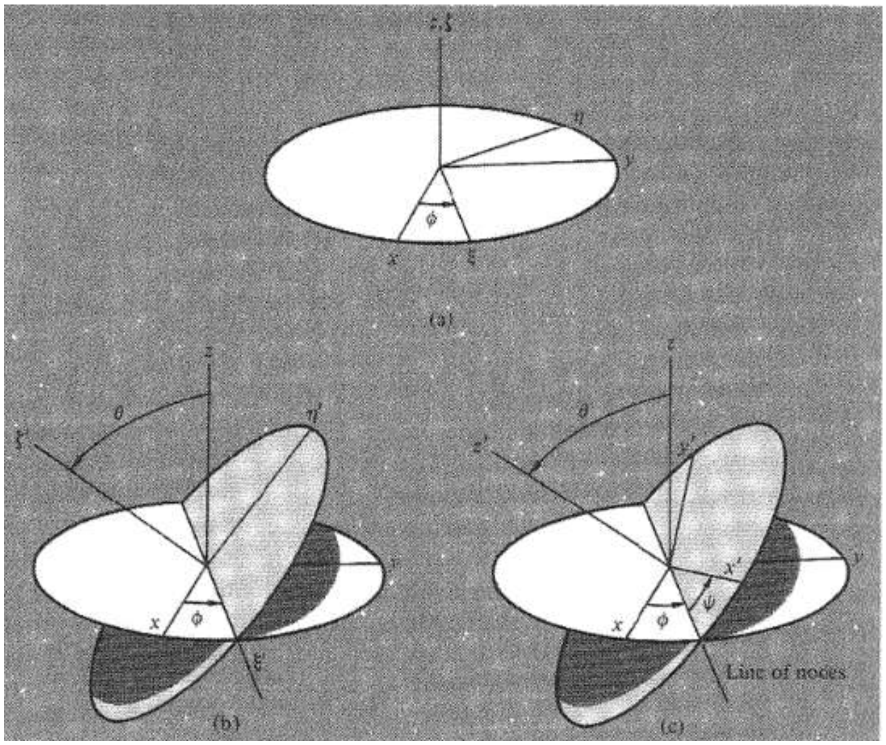

Figure 1. The rotations that define Eulerian angles [5]

The full transition from x to $x^{'}$ can be represented as

$$x^{'} = \boldsymbol{BCD}x.$$

The matrix multiplication $\boldsymbol{BCD}$ produces the matrix **A** such that

$$x^{'} = \boldsymbol{A}x.$$

with the matrix $\boldsymbol{A}$ defined as

$$\boldsymbol{A} = \begin{bmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi & \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}.$$

The overall angular velocity $\boldsymbol{\omega}$ can be defined as the sum of the individual velocities $\omega_\phi, \omega_\theta,$ and ω_ψ. These angular velocities are not all in terms of the $x'y'z'$ system, but the components of each of them can be rotated to express the total angular velocity in terms of the body system by using the matrices defined above. ω_ϕ is in the z direction, so it must be multiplied by $\boldsymbol{A}$. ω_θ is in the ξ direction which is the same as the ξ' axis, and therefore, only needs to be multiplied by $\boldsymbol{B}$. ω_ψ is already in the z' direction, so no transformation is necessary. The sum of the vectors given by these transformations will produce equations for the x', y', and z' components of ω_b, the total angular velocity of the body system.

The vectors for $\omega_\phi, \omega_\theta,$ and ω_ψ can be written as

$$\boldsymbol{\omega_\phi} = \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

$$\boldsymbol{\omega_\theta} = \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{\omega_\psi} = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}.$$

So,

$$(\boldsymbol{\omega_b})_{\boldsymbol{\phi}} = \boldsymbol{A\omega_\phi}$$

and

$$\boldsymbol{A\omega_\phi} = \begin{bmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi & \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}.$$

Therefore,

$$(\boldsymbol{\omega_b})_{\boldsymbol{\phi}} = \begin{bmatrix} \dot{\phi}\sin\psi\sin\theta \\ \dot{\phi}\cos\psi\sin\theta \\ \dot{\phi}\cos\theta \end{bmatrix}.$$

For the second transformation,

$$(\boldsymbol{\omega_b})_{\boldsymbol{\theta}} = \boldsymbol{B}\boldsymbol{\omega_\theta}$$

and

$$\boldsymbol{B}\boldsymbol{\omega_\theta} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix}.$$

So,

$$(\boldsymbol{\omega_b})_{\boldsymbol{\theta}} = \begin{bmatrix} \dot{\theta}\cos\psi \\ -\dot{\theta}\sin\psi \\ 0 \end{bmatrix}.$$

Because ω_ψ is already in the $z^{'}$ direction,

$$(\boldsymbol{\omega_b})_{\boldsymbol{\psi}} = \boldsymbol{\omega_\psi} = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}.$$

Thus, the total angular velocity with respect to the axes of the body system is

$$\boldsymbol{\omega_b} = (\boldsymbol{\omega_b})_{\boldsymbol{\phi}} + (\boldsymbol{\omega_b})_{\boldsymbol{\theta}} + (\boldsymbol{\omega_b})_{\boldsymbol{\psi}}.$$

Substituting in the matrices for $(\boldsymbol{\omega_b})_{\boldsymbol{\phi}}$, $(\boldsymbol{\omega_b})_{\boldsymbol{\theta}}$, and $(\boldsymbol{\omega_b})_{\boldsymbol{\psi}}$,

$$\boldsymbol{\omega_b} = \begin{bmatrix} \dot{\phi}\sin\psi\sin\theta \\ \dot{\phi}\cos\psi\sin\theta \\ \dot{\phi}\cos\theta \end{bmatrix} + \begin{bmatrix} \dot{\theta}\cos\psi \\ -\dot{\theta}\sin\psi \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi \\ \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{bmatrix}.$$

Therefore,

$$(\boldsymbol{\omega_b})_{x^{'}} = \dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi$$

$$(\boldsymbol{\omega_b})_{y^{'}} = \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi$$

$$(\boldsymbol{\omega_b})_{y^{'}} = \dot{\phi}\cos\theta + \dot{\psi}.$$

With respect to these velocities, the Lagrangian for the system is

$$L = T - V$$

where T is the kinetic energy from the processional, tipping, and rotational motion of the top with respect to each of the three body axes, $x^{'}, y^{'}$, and $z^{'}$, and V is the potential energy due to gravity working on the system. Therefore,

$$L = \frac{1}{2}I_1(\omega_b)^2_{x^{'}} + \frac{1}{2}I_2(\omega_b)^2_{y^{'}} + \frac{1}{2}I_3(\omega_b)^2_{z^{'}} - Mgh.$$

Remembering the definition of the Kovalevskaya top, $I_1 = I_2 = 2I_3$, and $I_3 = 1$ which means that

$$L = (\omega_b)^2_{x^{'}} + (\omega_b)^2_{y^{'}} + \frac{1}{2}(\omega_b)^2_{z^{'}} - Mgh.$$

The height of the center of mass at any given point in the top's movement corresponds to the tipping angle, so

$$h = l\cos\theta$$

where l is the distance from the bottom tip of the top to the position of its center of mass along the $z^{'}$ axis, and

$$L = (\omega_b)^2_{x^{'}} + (\omega_b)^2_{y^{'}} + \frac{1}{2}(\omega_b)^2_{z^{'}} - Mgl\cos\theta.$$

Plugging in the values for the $x^{'}, y^{'}$, and $z^{'}$ components of ω_b,

$$L = (\dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi)^2 + (\dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi)^2 + \frac{1}{2}(\dot{\phi}\cos\theta + \dot{\psi})^2 - Mgl\cos\theta$$

which simplifies to

$$L = \dot{\theta}^2 + \dot{\phi}^2\sin(\theta)^2 + \frac{1}{2}(\dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta + \dot{\phi}^2\cos(\theta)^2) - Mgl\cos\theta.$$

This form of the Lagrangian can now be used to find the equations of motion using the Hamiltonian Variational Principle.

Remembering the equation $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$, solve for $q = \phi, \theta,$ and ψ. The resulting equations will be equations of motion for Kovalevskaya's top. Starting with $q = \phi$, one must find $\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}}$ and $\frac{\partial L}{\partial \phi}$.

$$\frac{\partial L}{\partial \dot{\phi}} = 2\dot{\phi}\sin(\theta)^2 + \dot{\phi}\cos(\theta)^2 + \dot{\psi}\cos\theta,$$

so

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = 2\ddot{\phi}\sin(\theta)^2 + 4\dot{\phi}\dot{\theta}\sin\theta\cos\theta + \ddot{\phi}\cos(\theta)^2 - 2\dot{\phi}\dot{\theta}\cos\theta\sin\theta + \ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta.$$

Also,

$$\frac{\partial L}{\partial \phi} = 0.$$

Thus,

$$\ddot{\phi} = -\frac{4\dot{\phi}\dot{\theta}\sin\theta\cos\theta + \ddot{\phi}\cos(\theta)^2 - 2\dot{\phi}\dot{\theta}\cos\theta\sin\theta + \ddot{\psi}\cos\theta - \dot{\psi}\dot{\theta}\sin\theta}{2\sin(\theta)^2}.$$

Solving for $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}$ and $\frac{\partial L}{\partial \theta}$,

$$\frac{\partial L}{\partial \dot{\theta}} = 2\dot{\theta},$$

so

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 2\ddot{\theta},$$

and

$$\frac{\partial L}{\partial \theta} = \dot{\phi}^2\sin(\theta)\cos(\theta) + \left(Mgl - \dot{\phi}\dot{\psi}\right)\sin(\theta).$$

Thus,

$$\ddot{\theta} = \frac{\dot{\phi}^2\sin(\theta)\cos(\theta) + \left(Mgl - \dot{\phi}\dot{\psi}\right)\sin(\theta)}{2}.$$

Finally, when $q = \psi$,

$$\frac{\partial L}{\partial \dot{\psi}} = \dot{\psi} + \dot{\phi} \cos \theta,$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\psi}} = \ddot{\psi} + \ddot{\phi} \cos \theta - \dot{\phi}\dot{\theta} \sin \theta,$$

and

$$\frac{\partial L}{\partial \psi} = 0.$$

Thus,

$$\ddot{\psi} = \dot{\phi}\dot{\theta} \sin \theta - \ddot{\phi} \cos \theta.$$

The equations for $\ddot{\phi}, \ddot{\theta},$ and $\ddot{\psi}$ define the motion of the top with respect to the Eulerian angles of the body system. Kovalevskaya's solution, which included the integration of hyperelliptic quadratures and abelian functions, was so beautiful that the award for her work was increased to match the caliber of work done.

Three years after receiving the Prix Bordin for her work on the rotation of a rigid body around a fixed point Kovalevskaya died of pneumonia at the age of 41. While she was never able to achieve her ultimate goal of teaching at a university in Russia, she was given a position as a corresponding member of the Imperial Academy of Sciences. Kovalevskaya died at the peak of her mathematical journey, having left an undeniable mark on the European mathematics community.

Germain and Kovalevskaya gave themselves wholeheartedly to their pursuits of mathematics, fighting for their places in mathematical history, and their efforts were rewarded despite the views of their societies. Both Sophie Germain and Sonya Kovalevskaya greatly influenced mathematics through their own work and collaborations with their mathematical contemporaries. Though neither was able to realize their goals fully because of societal prejudice both inside and outside academia, they did make the most of the opportunities that they were able to find. Today, mathematical concepts and theorems carry their names, and though their stories may not be well known, these women led extraordinary lives that prove that women have a place in mathematics.

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