Probability to 1750

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Introduction

Probability as a mathematic discipline is fairly young when compared to algebra, number theory, geometry, trigonometry and arithmetic. This paper will trace the inception of probability theory in Western Europe from antiquity to its estimated "birth" in the mid-1600s, and through its early infancy during the following century. The time period for this work was chosen due to the fact that "[within] the decade surrounding 1660…probability entered European thought and then it happened in two senses; first as a way of understanding stable frequencies in chance processes, and then as a method of determining reasonable degrees of belief" (Katz, 1998, p 451).

This paper will focus only on the history of probability up to approximately 1750. It is important to note that most of the mathematicians introduced did not limit their attention to probability and may have been great contributors to other disciplines as well.

We begin with a definition of probability, and then introduce the problem of points and the mathematicians who identified the problem and began work on a solution to it. This focus on the issue of the division of stakes in a fair game will be followed by the seemingly natural progression from expected results in gaming to calculating the probability of events in general. We see the problem of points again when we conclude with de Moivre's *Doctrine of Chances*.

Due to the short length of this work, there will be much about each mathematician that will be left to the reader to discover; however, the expectation is that the information contained herein will give a relatively thorough overview of the history of probability to 1750.

What is Probability?

Merriam-Webster's defines probability as "the chance that something will happen" or "a measure of how often a particular event will happen if something (such as tossing a coin) is done repeatedly". This definition, while satisfactory to some, leaves much to be desired. The research conducted for this project did not reveal a single universally agreed-upon definition of the discipline. In fact, probability itself can be thought of as an ambiguous concept in which the meaning of the word is filtered through the schemata of the definer. We consider here the definitions offered by two experts in the history of probability, both of whom regard the meaning of "probability" as two-fold. The first, given by Ian Hacking in *The Emergence of Probability* (1975), presents two aspects: it is "…connected with the degree of belief warranted by evidences" while its other aspect is that "it is connected with the tendency, displayed by some chance devices, to produce stable relative frequencies" (p 1).

The second definition is offered by Anders Hald in *A History of Probability and Statistics and Their Applications before 1750* (1990) in which he focuses on what he feels are the two major contexts in which probability is found. He says "*Objective, statistical, or aleatory* (dependent upon chance) *probabilities* are used for describing properties of random mechanisms or experiments…and for describing chance events in populations", whereas "*Subjective, personal, or epistemic probabilities* are used for measuring the degree of belief in a proposition warranted by evidence which need not be of a statistical nature" (p 28). While the two authors word their definitions differently, the essence of the meaning is the same: that probability can be thought of as the objective calculation and analysis of tendencies and frequencies, or as the subjective measurement of the degree of belief. This work will focus primarily on the objective context and the calculation of tendencies.

Probability from antiquity to the Renaissance (1350-1600)

The concept of chance in both gambling and rule-making can be traced back before the birth of Christ. In *Games, Gods and Gambling* (1962), F.N. David observes that there is little surviving evidence to determine when gambling became part of the human experience, but discusses the fact that the Greeks may have arrived at their games of chance through contact with Egyptians before 3500 B.C. David (p. 7) goes on to say that, "In yet another commentary we are told that Palamedes invented games of chance during the Trojan wars". In any case, by the time of the Roman Empire, gaming was a common form of recreation for the people.

Gambling was not the only use of probability in antiquity as evidenced in Jewish texts such as the Talmud and rabbinic literature. Fortune-telling, games of chance, philosophy, law, and insurance all have historic roots in probability. Prior to the Renaissance, European mathematics was at a virtual standstill, with very few new methods, discoveries or interpretations of prior results being made. The early medieval period was a time in which the Church exerted a great influence on all aspects of society. It's not surprising, then, that much of the intellectual energy of the time was spent on spiritual and theological questions as opposed to a focus on mathematics and science. Prior to the period of the Renaissance, probability was still a non-numerical concept; only in the Renaissance was chance expressed as a ratio, and a calculus of chances became a part of algebra.

Interestingly, as Europe moved into the High Middle Ages, the question of rules in gaming became an impetus for change. As the region began to recover from the effects of the plague, society began to turn its attention to a revival of the Greek model of classical learning and a landslide of intellectual progress began

As noted by David (1962), this era was a turning point not only in probability, but in all intellectual pursuits. David goes on to aptly describe the fundamental shift that occurred in this period as follows:

> The difference in spirit between classical times and the end of the Dark Ages is remarkable…With the renaissance of the human intellect there is the beginning of the empirical tradition, from observation to hypothesis and back to observation again, which has led to the great strides of modern science (p 35).

As Europe recovered from the Black Death, there was a renewed focus on literature, sculpture and painting, with advances in science and mathematics made as a result of the revival of classical scholarship.

In addition to advancements in probability theory, the Renaissance and early 17th century mathematicians produced and proved theories in combinatorics, infinite series and analysis and made great strides in the beginnings of calculus. New methods of proof were also being created and tested while societies of scientists and mathematicians shared the new wealth of information, working together (and sometimes in great opposition) to the intellectual benefit of humanity.

16th Century and the Division problem

By the 1500s, games of chance and lotteries took a more prominent place in European society. In fact, lotteries were increasingly being used to fund government expenditures. While both the Roman Catholic and Protestant Churches frowned on gambling, the practice grew in popularity, perhaps due in part to the government's usage to increase their treasuries. In fact, the first recorded British Lottery was charted in 1656 by Queen Elizabeth I. It is not surprising then, that mathematicians sought to describe games of chance in terms of the odds of winning as well as to analyze the division of stakes in a fair game.

The problem of stakes in a fair game is also known as the *division problem* or the *problem of points* and it is this: How do players in a game of chance divide the stakes in a game that had been interrupted or otherwise prematurely terminated? The division problem is mentioned in writings as early as 1494. In fact, a Latin poem, *De Vetula*, was written sometime between 1200 and 1400 that gives an analysis of the different ways in which three dice can be rolled.

Consider this situation: Suppose two players agree to play a series of 7 games, but are interrupted after the first player has won 3 games and the second has won 2. How, then, should they divide the stakes?

Girolamo Cardano (1501-1576)

While attempts had been made to solve this problem previously, one of the more colorful mathematicians of the time, Girolamo Cardano is credited with one of the earliest treatises on the issue. Cardano was an Italian mathematician, physician, astrologer, philosopher and inveterate gambler. His body of written work is extensive with over 200 works published in his lifetime. It was his desire that his vast knowledge be shared with the world.

His manuscript, *Liber de Luda Aleae* (Book on games of chance), was written in the year 1564 and published after his death. Cardano intended *De Ludo Aleae* can be considered a treatise on the moral, practical, and theoretical aspects of gambling and is considered to be a work of lasting import in both philosophy and probability theory. As noted by Hald, "Most of the theory in the book is given in the form of examples from which general principles are or may be inferred" (p 38). One of the general principles that Cardano does give is the following as quoted by Hald (p. 38):

> So there is one general rule, namely, that we should consider the whole circuit, [the total number of equally possible cases], and the number of those casts which represents in how many ways the favorable result can occur, and compare that number to the remainder of the circuit, and according to that proportion should the mutual wager be laid so that one may contend on equal terms (p 38).

Among Cardano's results is the application of the addition rule as well as the multiplication rule for finding an expected value for future plays. Using modern notation, Hald translated one such rule: "Let the number of equally possible cases in the game be t, and let r be the number of favorable cases, so that the odds are $r/(t-r)$…in n repetitions, the odds will be $r^n/(t^n-r^n)$. Setting $p = r/t$, the result becomes $p^n/(1-p^n)$, which is the form used today" (Hald, 1990, p39).

17th Century

Travel, trade and communication with other cultures brought Hindu number-writing methods and Islamic algebraic methods to Europe in the 15th century. By the seventeenth century, Western European mathematicians, including Cardano, had developed these methods further, introduced symbolism into algebra and continued the application of the arithmetic triangle and the calculation and use of binomial coefficients. "Throughout these centuries men continued to gamble with both cards and dice, and somehow among mathematicians the knowledge of how to calculate probabilities percolated" (David, 1962, p 62). From the time of Cardano's *Liber de Luda* there was little written mention of probability from the mathematical community, though as David (1962) suggests, the division problem was still of some interest to society in general and to mathematicians in particular. David further suggests the possibility that gaming problems were discussed at meetings of the various learning societies that were being established during this period (p 63).

Beginning in the 17th century, an increasing number of mathematicians wrote specifically on problems involving probability. We now turn to a description of the most important of these contributions, beginning with those of Galileo-Galilei.

Galileo-Galilei (1654-1742)

Galileo Galilei was an Italian professor of mathematics whose name is synonymous with physics and astronomy. His contributions to probability theory are not as well known, but may be considered quite important. Sometime between 1613 and 1623, Galileo focused his attention on the following problem:

> Playing with three dice, 9 and 10 points may each be obtained in six ways. How is this fact compatible with the experience that long observation has made dice-players consider 10 to be more advantageous than 9? (Hald, 1990, p 41).

Galileo's analysis of the problem was published in *Sopra le Scopetre dei Dadi* (On a discovery concerning dice, 1898). He pointed out that "The fact that in a dice game certain numbers are more advantageous than others has a very obvious reason, i.e. that some are more easily and more frequently made than others" (Galileo, 1898, pp 591-4). For example, when using 3 dice, triples that are made of three equal numbers can only be made in one way. On the other hand, any triple that included three different numbers (e.g. 1,3,6) can be made up in 6 different ways (e.g. 136, 163, 316, 361, 613, 631). Galileo discusses this idea in some detail and then gives a table to summarize all of the possibilities.

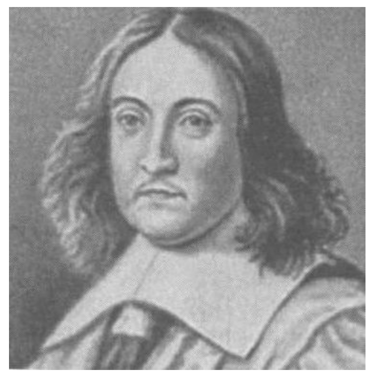
Galileo describes the following table by indicating that the top row are the points of the throws from 10 down to 3, followed below by the different triples that form these results. Note that the leftmost column was created by adding the number of ways that one could obtain the sums from 3 to 10 and then doubling that total of 108 to account for the number of ways one can obtain sums of 11-17:

1 3 6	10		9		8		7		6		5		4		3	
10 15 21 25 27	631 622 541 532 442 433	6 3 6 6 3 3	621 531 522 441 432 333	6 6 3 3 6 1	611 521 431 422 332	3 6 6 3 3	511 421 331 322	3 6 3 3	411 321 222	3 6 1	311 221	3 3	211	3	111	1
108		27		25		21		15		10		6		3		1

Galileo's work is "remarkable not only because of the recorded observation…but also because somebody…asked for and actually got an explanation in terms of a probabilistic model" (Hald, 1990, p 41). Prior to *Sopra le Scoperte dei Dadi*, works by other mathematicians, including Cardano's, referred to any empirical data. While Galileo did not solve the problem of points, his was the first record of examples of the possible outcomes.

Blaise Pascal (1623-1662) and Pierre de Fermat (1601-1665)

Blaise Pascal



Pierre de Fermat

Blaise Pascal, known for both his decision-theoretic argument for belief in God (or Pascal's Wager[1]) and for his version of the arithmetic triangle, was one of the two most arguably important mathematicians in the history of probability. The other figure is Pierre de Fermat who, in addition to his contribution to probability, holds an important place in mathematic history due to his work in calculus, analytic geometry and number theory.

"According to legend probability began in 1654 when Pascal solved two problems and then wrote to Fermat" (Hacking, 1975, p 11). Hald (1990) is even more emphatic about the contributions of the two mathematicians when he says of Pascal, "Together with Fermat he laid the foundation of probability theory in 1654, and at the same time he wrote the important treatise on the arithmetic triangle" (p 44).

In 1654 the two began a correspondence which has since been translated into English and can be found in *A Source Book in Mathematics*. The problem of points was posted to the two mathematicians

[1] In his *Pensées*, Pascal combines probability theory, decision theory, and theism to argue that if God does not exist and one does not believe in him, one gains nothing. If God does exist and one does not believe, then one loses any eternal reward. The "wager" one makes is with one's own eternal soul.

by the Chevalier de Méré, "a gambler who is said to have had an unusual ability 'even for the mathematics'" (Smith, 1959, pg 546). During this era, there were several societies of learned groups that met and shared information. Additionally, since scientific journals did not exist in the form we know them today; those who could not easily meet in person to discuss their ideas did so in the form of written correspondence.

Thus, the division of stakes is discussed between the two mathematicians in a series of letters and the results of their work were shared with their colleagues, perhaps at one of the society's gatherings. While the initial letter from Pascal to Fermat is no longer available, the remaining portions of the correspondence are a first-person account of one of the foundations of probability theory. Given the foundational nature of the correspondence, we now consider its contents in some detail.

The first of the remaining correspondence begins with Fermat's letter to Pascal , in which he argues that the way to calculate the value of the shares when two gamblers play is "…given any number of throws that one wishes, to find the value of the first." He says that "…if each one plays the number of pistols[2] expressed by the product of the even numbers, there will belong to him [who forfeits the throw] the amount of the other's wager expressed by the product of the odd numbers" (549).

From Pascal's response, dated Monday, August 24, 1654 it is obvious that Pascal does not agree entirely with Fermat's solution, but is very careful to keep his remarks as innocuous as possible. In fact, he begins by saying that he is reluctant to tell Fermat his thoughts "…lest this admirable harmony which obtains between us and which is so dear to me should begin to flag…" but goes on to say, "I wish to lay my whole reasoning before you, and to have you do me the favor to set me straight if I am in error or to indorse me if I am correct" (554). The problem Pascal has with Fermat's initial solution is that, while the combinations used for two players is essentially correct, the same method cannot be extended to more than two players without losing the "fairness" that the two sought to maintain. Pascal then introduces the number of combinations possible when there are three players, listing the sample space and highlighting the fact that with the ensuing probabilities, the proportion of the stake that each player will be entitled to amounts to 19, 7 and 7 for players 1, 2 and 3, respectively. He points out that this method is not equitable.

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[2] Pistols are a monetary unit of the time.

> It seems to me that this is the way in which it is necessary to make the division by combinations according to your method, unless you have something else on the subject which I do not know. But if I am not mistaken, this division is unjust. (559)

Pascal argues that the error lay in the fact that they had made a false supposition, supposing that there were an exact amount of plays that would remain once it was obvious that the game would not continue to its natural conclusion but was interrupted in some way. He concludes this letter by stating that Fermat's method was correct in the case of two players, but possibly only accidentally, and then asks Fermat to keep him informed as he proceeds with his research on the problem.

Fermat's reply on August 29, 1654, indicates a different feeling about the conversation, at least from his point of view when he begins, "Our interchange of blows still continues, and I am well pleased that our thoughts are in such complete adjustment as it seems since they have taken the same direction and followed the same road" (560). Fermat offers praise to Pascal on his *Traité du triangle arithmétique* (1654), then says that he is "persuaded that the true way to escape failure is by concurring with you" (561).

It is interesting to note that Pascal and Fermat's correspondence was not limited to probability theory, but the sharing of Fermat's work in number theory and Pascal's Triangle, a triangular array of binomial coefficients whose roots can be traced back to ancient mathematicians.

Fermat wrote again to Pascal on September 25, 1654, continuing to refine their solution to the problem of points, highlighting that the sum of the chances for the first player to win as 1/3, 2/9, and 2/27, making 17/27 and thus building on the combinations he originally introduced, then revised due to Pascal's contributions. He concludes that

> This rule is good and general in all cases of the type where, without recurring to assumed conditions, the true combinations of each number of throws give the solution and make plain what I said at the outset that the extension to a certain number of points is nothing else than the reduction of divers fractions to the same denomination. Here in a few words is the whole of the mystery, which reconciles us without doubt although each of us sought only reason and truth. (563)

The final letter, from Pascal to Fermat, on October 27, 1654, Pascal tells Fermat that he admires the method and indicates that there is, once again, "harmony" between them. He tells Fermat that he has shared their solution (though he indicates it should belong to Fermat) and it was well received, saying, "All of our gentlemen saw it on Saturday last and appreciate it most heartily. One cannot often hope for things that are so fine and so desirable" (565), thus ending their correspondence.

One of the most important results of their correspondence was the recognition of probability as a worthy mathematical pursuit. In an address to the Academie Parisienne de Mathematiques in 1654, Pascal is quoted in Hald (1990):

> Thus, joining the rigour of demonstrations in mathematics with the uncertainty of chance, and conciliating these apparently contradictory matters, it can, taking its name from both of them, with justice arrogate the stupefying name: The Mathematics of Chance. (p 42)

Christiaan Huygens (1629-1695)

Christiaan Huygens was a Dutch mathematician and scientist. He is known primarily for his work in physics and astronomy and is credited with inventing the pendulum clock. Huygens was familiar with the problem of points, but may have been told of the solutions of Fermat and Pascal through his communications with Marin Mersenne, a French theologian, mathematician and philosopher who formed the Academie Parisienne . (David, 1962) Huygens derived three theorems on expectation based on an axiom on the value of a fair game and published them in 1657 in The *De Ratiociniis in Ludo Aleae*. "This treatise…was, it is said, warmly received by contemporary mathematicians, and for nearly half a century it was the unique introduction to the theory of probability" (David, p 115).

Huygen's first three propositions are:

1. "If I have equal chances of getting *a,b or c,* this is so much worth for me as (*a+b*)/2".
2. "If I have equal chances of getting *a,b* or *c*, this is so much worth for me as if I had (*a+b+c*)/3."
3. "If the number chances of getting *a* is *p,* and the number of chances of getting *b* is *q*, assuming always that any chance occurs equally easy, then this is worth (*pa+qb*)/(*p+q*) to me."

In these propositions, we can see the very definition of mathematical expectation.

Huygens, then, had invented a new method for solving problems of points when there is no upper limit to the number of games.

James Bernoulli (1654-1705)

James Bernoulli (also known as Jakob or Jacques) was of Dutch origin who came from a family of mathematicians. He became the chair of mathematics at a university in Switzerland after his family moved from Holland. He is known for his work in calculus and probability.

According to Hald (p.221), James Bernoulli, along with his nephew Nicholas, began corresponding regularly with Huygens in the late 1700s, though the majority of their discussions were centered on differentiation and integration. This contrasts with David's account in which James Bernoulli was greatly influenced by Huygen's work, but did not correspond with him directly (p 132). In any event, James Bernoulli became interested in the calculus of probability and wrote his *Meditationes* between 1684 and 1690, beginning by explaining his solutions to the problems Huygens put forth in *De Ratiociniis in Ludo Aleae*.

When Bernoulli died, among his papers was found a manuscript that has become quite celebrated in the history of probability; *Ars Conjectandi* (the Art of Conjecturing). Due to bad blood between James' widow and his surviving family, the work was not published until 1713, five years after his death. According to the history scholars used as sources here, *Ars Conjectandi* was not complete at the time of his death and it's unknown exactly when the manuscript was written.

David (1962) claims that "It seems likely that James had some idea of using at least part of the *Ars Conjectandi* as a textbook for his students since he is concerned not only to give correct solutions but on occasion to point out errors in reasoning which may lead to wrong answers" (p 135).

The following description of the contents of *Ars Conjectandi* is drawn primarily on the work of Hald(1990):

1) The treatise *De Ratiociniis in Ludo Aleae* by Huygens with annotations by James Bernoulli.

2) The doctrine of permutations and combinations.
3) The use of the preceding doctrines on various games of chance and dice games.
4) The use and application of the preceding doctrines on civil, moral and economic affairs.

The first three sections of Bernoulli's great work focus on the results obtained by Pascal and Huygens as they relate to the problem of points. He uses the arithmetic triangle and derives the binomial distribution[3], then goes on to find the probability of winning a game with a possibly infinite number of trials by taking the sum of an infinite series.

In the fourth and final part of the work, Bernoulli advances new problems and applications of theory, redefining probability as "a measure of our knowledge of the truth of a proposition" (Hald, p 225). One of the reasons that *Ars Conjectandi* was, and remains, such an important work in probability theory was the fact that he not only gave and solved example problems, he also generalized results and provided new methods for their solutions. Perhaps one of Bernoulli's most important contributions is the introduction of the term "permutation" into combinatorial theory.

Another aspect of Bernoulli's work that is intriguing is that fact that he is the first to distinguish between a priori (deductively obtained) and posteriori (inductively from relative frequencies) probabilities. As probability was used in other fields such as life insurance, Bernoulli felt it was important to be able to calculate the frequency of an event by first identifying the number of equally likely outcomes. In order to do this, he says that "the more observations that are taken, the less the danger will be of deviating from the truth" (Hald, p 257). To this end, Bernoulli theorized that, in a trial with $t=r+s$ equally likely outcomes, with r designated as the favorable outcome, then $p=r/(r+s)$. He then proves that, if n is large, the expected outcome will not deviate very much from the true probability of the event. His theorem, as presented by Hald (1990, pp 260-61), and is now known as The Law of Large Numbers, is shown here in modern notation:

Bernoulli's Theorem. Let r and s be two positive integers and set $p = r/(r+s)$ and $t = r+s$. For any positive real number c, we have

$$Pr\left\{|h_n - p| \leq \frac{1}{t}\right\} > \frac{c}{c+1}$$

[3] The binomial distribution is a discrete probability distribution of the number of successes and failures in a "yes" or "no" experiment of n trials. This distribution yields a probability p for the number of expected "yesses". In the case where $n = 1$, this is known as the Bernoulli distribution.

For $n = kt$ sufficiently large, i.e., for $k(r,s,t) \geq k(r,s,c) \vee k(s,r,c)$, where $k(r,s,c)$is the smallest positive integer satisfying

$$k(r,s,t) \geq \frac{m(r+s+1)-s}{r+!},$$

and m is the smallest positive integer satisfying

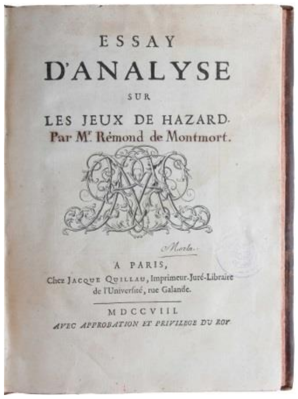
$$m \geq \frac{\ln[c(s-1)]}{\ln[(r+1)/r]}.$$

Also included in the fourth section of his work, Bernoulli introduces the concept of what he called "moral certainty". His view was that "for an outcome to be morally certain, it should have a probability no less than 0.999. Conversely, an outcome with a probability no greater than 0.001 he considered to be morally impossible" (Katz, 1998, p 599). Since Bernoulli was interested in determining whether an event was morally certain, he needed to formulate a way to determine the true probability of an event. It was this idea, then, that led to this Law of Large Numbers.

The importance of *Ars Conjectandi* to the development of probability cannot be denied. Interestingly, 2013 marks the 300th anniversary of its publication and much recent work has been written on its continued importance to modern probability theory. One such paper, by Edward Waymire, is published in the Bulletin of the American Mathematical Society:

> In particular, in recognizing the publication of *Ars Conjectandi*, we are celebrating an important symbol of what can be achieved through observation, data, and mathematical thinking. The seemingly simple quantity $P(f(X) > a)$, viewed in its broadest interpretations, embodies a quantification of uncertainty that lies at the heart of an immense span of deep mathematical theory and application. Therein, one finds a power, a beauty, and an intrigue of probability and statistics. The scope is immeasurable.

Pierre Rèmond de Montmort (1678-1719)

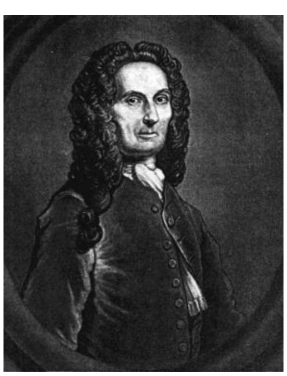


Pierre de Montmort was a contemporary of John and Nicholas Bernoulli, with whom he corresponded for many years. He was known for his work in probability and may have been the first to call Pascal's work on the arithmetic triangle Pascal's Triangle. While Montmort's name is not as well-known as many of the others who had a part in the development of probability theory, his contributions should not be omitted. In 1708, he published his *Essay d' Analyse sur les Jeux de Hazard* (Analytical Essay on Games of Chance). Montmort's Essay was the first published comprehensive text on probability. Comprised of three parts devoted to card games, dice games, and some additional problems involving games of chance, it combined the work of Pascal, Huygens and (James) Bernoulli.. According to Hald (1990), "Montmort continues in a masterly way the work of Pascal on combinatorics and its application to the solution of problems on games of chance. He also makes effective use of the methods discussed by Huygens. Finally, he uses the method of infinite series, as indicated by Bernoulli" (p 291).

The *Essay* may well have been the impetus for further advances to be made by Nicholas Bernoulli and de Moivre. That Montmort's work, while not groundbreaking, paved the way for the future of probability theory, is summarized as well by David (1962):

> Montmort's importance from the probability point of view is possibly not in the new ideas which he introduced but in the algebraic methods of attack. These were perhaps much the same as those of James Bernoulli, but the two mathematicians, among them de Moivre, who would not have been interested solely in the laborious enumeration of the fundamental probability set. (p 160)

Abraham de Moivre (1667-1754)



Abraham de Moivre was a French mathematician who was a close friend of Isaac Newton and is credited with the development of trigonometry who made a living as a consultant for insurance and gambling concerns.

As both David and Hald mention, Montmort's work in probability led to additional work by other mathematicians. Of particular interest is the work of Abraham de Moivre, who, after reading Montmort's *Essay* "…realized that all the results on the problem of coincidences may be derived from a general theorem on the probability of compound events" (Hald, 1990, p 336). In 1711 de Moivre published *De Mensura Sortis, seu de Probabiltate Eventuum in Ludis a Casu Fortuito Pendtibus* in which he creates this generalization based upon a problem posed by Montmort.

The problem was this: given any number of letters *a,b,c,d,e,…,n*, where $a \neq b \neq c \neq d \neq e \neq \cdots \neq n$, "…to find the probability that some of them shall be found in their places according to the rank they obtain in the alphabet; and that others of them shall at the same time be displace" (de Moivre, London, 1756 P 230 as quoted by Hacking). His generalization on the probability of at least one coincidence is given by Hald (1990) on page 340:

$$\sum_{i=1}^{n} \frac{(-1)^{i-1}}{i!\,(n^{i!})^{m-2}}$$

An enlarged version of *De Sortis*, de Moivre's famous *Doctrine of Chances,* was published in 1718 with revisions published in 1738 and 1756.

In the introduction to the *Doctrine*, de Moivre writes that "The General Theorem invented by Sir Isaac Newton, for raising a Binomial to any Power given, facilitates infinitely the Method of Combinations, representing in one View the Combination of all the Chances, that can happen in any given Number of Times. (de Moivre, London, 1756 p xix).

While much progress had been made in probability since the correspondence between Fermat and Pascal, the problem of points still figured largely in the mathematics of chance. This is evident in de Moivre's *Doctrine* when, while discussing two gambler of unequal skill, he says:

> It is true that this degree of skill is not to be known any other way than from Observation : but if the same Observation constantly recur, 'tis strongly to be presumed that a near Estimation of it may be made… (as quoted in David, 1962, p 168)

With his approximation to the individual terms of the binomial expansion, de Moivre was able to calculate the sums of large numbers of such terms and thus improve considerably on Bernoulli's quantification of uncertainty.

Conclusion

From antiquity to the 18^{th} century, great advancements were made in the mathematics of chance. From the division of stakes in a prematurely interrupted game with just two players, to the calculation of expected outcomes using infinite series and integration, probability had come to be recognized as an intellectual pursuit worthy of the best minds of the time. The work of Cardano, Galileo, Pascal, Fermat, Bernoulli, Montmort and de Moivre was soon followed by advances in statistical inference by Laplace and conditional probability by Bayes.

Pascal's Triangle, Bernoulli's Law of Large Numbers and de Moivre's Normal approximation of the binomial distribution are still in use today. Their calculations, derivations and proofs have maintained their importance over the last 300 years. What began as a discussion of the probability of winning a dice game became, in a relatively short time, a process by which expectation of events in almost any discipline could be calculated and interpreted.

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