Paul Ayers

Gabriel Cramer: Over 260 years of crushing the unknowns

University of Missouri – Kansas City

Supervising Instructor: Richard Delaware

pagw7@mail.umkc.edu

Postal address: 12522 S. Crest Drive, Olathe, KS 66061

Gabriel Cramer: Over 260 years of crushing the unknowns

Gabriel Cramer: Over 260 years of crushing the unknowns

In my study of mathematics, I have found the subject of linear algebra and the application of it to the field of Computer Science especially interesting. One of the main topics in linear algebra is solving the linear system $A\mathbf{x} = \mathbf{b}$ in which A is a known coefficient matrix, $\mathbf{b}$ is a known vector and $\mathbf{x}$ is an unknown vector; I use bold font for the letters $\mathbf{x}$ and $\mathbf{b}$ only to show they are vectors. For example, consider the following system:

$$\begin{cases} 2x_1 + x_2 - 5x_3 = 1 \\ 4x_1 + x_3 = 7 \\ x_1 - 3x_2 + 6x_3 = 4 \end{cases}$$

Matrix A is composed of the coefficients of the system so that the number of rows is equal to the number of equations, namely 3, the number of columns of A corresponds to the number of variables, again 3, vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is unknown and vector $\mathbf{b} = \begin{bmatrix} 1 \\ 7 \\ 4 \end{bmatrix}$ is the value given to each equation. Therefore, to rewrite the previous system in matrix notation, we have:

$$\underset{A}{\begin{bmatrix} 2 & 1 & -5 \\ 4 & 0 & 1 \\ 1 & -3 & 6 \end{bmatrix}} \underset{\mathbf{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\mathbf{b}}{\begin{bmatrix} 1 \\ 7 \\ 4 \end{bmatrix}}$$

The goal is to solve for the unknown vector $\mathbf{x}$ and there are many different ways to do so. The system can contain any number of variables and so any number of unknowns; therefore there are advantages to knowing the options available for solving it and which ones work best depending on the size. One of the early contributors to this subject was a man named Gabriel Cramer (1704 − 1752), whose name is attached to the popular Cramer's Rule which is frequently used in linear algebra for solving the system $A\mathbf{x} = \mathbf{b}$. In this paper I will discuss

Cramer's life and the work he did that led to Cramer's Rule and its application to linear algebra today.

Gabriel Cramer was born July 31, 1704 in Geneva, Switzerland and was one of three sons born to Jean Isaac and Anne Mallet Cramer (1,1). Gabriel Cramer's father was a medical doctor in Geneva, one of his brothers was also a medical doctor and the other was a professor of law, so he was no doubt well educated as a child. In fact, it was in 1722, when Cramer was only eighteen, that he submitted his thesis on the theory of sound and was presented with a doctorate for his work from the Académie de la Rive (1,2). A short two years later, Cramer was competing for the chair of philosophy at the Académie de Calvin in Geneva and while he did not win the chair it was still a victory because the magistrates were so impressed by Cramer and another young applicant that they created a chair of mathematics and appointed them both to it (1,2). It was during this time that Cramer began to travel and study internationally, writing on several topics including philosophy, agriculture and mathematics. Cramer seemed to have a broad spectrum of interests and remained an involved citizen in the areas in which he lived; it is said he was involved with artillery and fortification in one town and he also gave instructions to workers rebuilding a cathedral. Cramer never married in his life, but he was reported to be "friendly, good-humored, pleasant in voice and appearance, and possessed of good memory, judgment and health" (1,3).

Cramer taught his courses in French although Latin was the popular academic language of his time and he had the opportunity to work with some incredible mathematicians including Johann Bernoulli, Leonhard Euler, Edmond Halley and Colin Maclaurin among others (1,9).

Gabriel Cramer: Over 260 years of crushing the unknowns

Cramer's major mathematical publication *Introduction à l'analyse des lignes courbes algébriques* [Introduction to the analysis of algebraic curves] no doubt included work influenced by Cramer's gifted colleagues. Shortly after the publication of his book, Cramer was seriously injured during a fall from a carriage. During his recovery he decided to spend some time in the south of France at the advice of his doctor, however Cramer died on January 4, 1752 during the trip to southern France (1,5).

In Cramer's book *Introduction à l'analyse des lignes courbes algébriques*, he wrote on an array of subjects, but I will focus on the area of the book from which the algorithm came that currently we refer to as Cramer's Rule. Today Cramer's Rule solves a system of linear equations by way of the determinant which is a special number associated with a square coefficient matrix. Cramer's book however, was published shortly before determinants were invented; therefore Cramer's example is a little different than the modern version. Cramer's explanation of how to solve a linear system for n equations with n variables is actually found in the Appendix of his book on pages 656 – 659. Colin MacLaurin published a book in 1748 giving a proof for solving square linear systems with either two or three unknowns, however Cramer offers no formal proof in his book, but instead simply explains in a straightforward manner how to solve for the unknowns in a system of linear equations. I was fortunate enough to find an original copy of Cramer's book in the Linda Hall Library rare book room from which I translated pages 656 – 659 from French to English. My translation of these pages is what follows; I've included a few pictures of the original text in the following excerpt.

[Notes in brackets are my own words.]

Gabriel Cramer: Over 260 years of crushing the unknowns
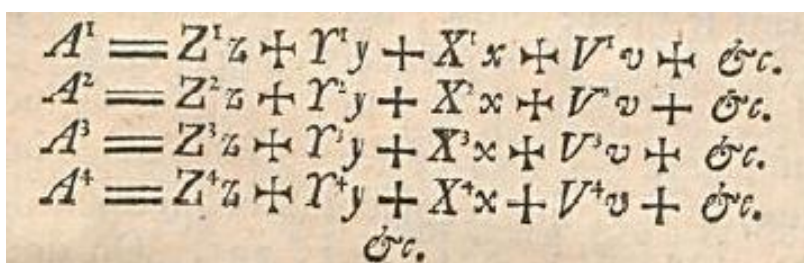
The vanishing of unknowns

When a problem contains several unknowns, whose relations are so complicated that it is required to form several equations; then, to find the values of these unknowns, we make them all vanish, the least [of which] when combined with the known quantities gives, if the problem is determined, a final equation whose solution is to know first the [least] unknown, and afterwards by its means all the others.

The Algebra of the Rules provides for this, whose success is infallible, provided you have the patience to follow [all of the calculations]. But the calculation becomes extremely long, when the number of equations and unknowns are great, and also when these unknowns are raised to high degrees in the proposed equations. In this second case we fall, by ordinary methods, [into] another disadvantage: which is being led to equations composed of more than is necessary, and which contain extraneous roots, it is not always easy to disentangle those that give the true solution to the problem. I propose in the two parts the following as a remedy to these drawbacks.

Part I

Given several unknowns z, *y, x, v, & c.* [here, "*&c.*" actually means "et cetera" which we abbreviate today as etc.] & as many equations:

$$
\begin{aligned}
A^1 &= Z^1 z + Y^1 y + X^1 x + V^1 v + \&c. \\
A^2 &= Z^2 z + Y^2 y + X^2 x + V^2 v + \&c. \\
A^3 &= Z^3 z + Y^3 y + X^3 x + V^3 v + \&c. \\
A^4 &= Z^4 z + Y^4 y + X^4 x + V^4 v + \&c.
\end{aligned}
$$
$$\&c.$$

Gabriel Cramer: Over 260 years of crushing the unknowns

where the given letters $A^1, A^2, A^3, A^4, \&c.$ do not mark, as usual, the powers of A, but

the first member, assumed to be known, of the first, second, third, fourth $\&c.$ equation.

In the same way $Z^1, Z^2, \&c.$ are the coefficients of z; $Y^1, Y^2, \&c.$ are those of y;

$X^1, X^2, \&c.$ are those of x; $V^1, V^2, \&c.$ are those of v; $\&c.$ in the first, second, $\&c.$

equation.

This notation supposed, if there is an [one] equation & an [one] unknown z, we

will have [one of each, namely $A^1 = Z^1 z$ i.e.,] $z = \dfrac{A^1}{Z^1}$. If there are two equations and two

unknowns z & y, we [have $A^1 = Z^1 z + Y^1 y$ and $A^2 = Z^2 z + Y^2 y$ from which we] find

$z = \dfrac{A^1 Y^2 - A^2 Y^1}{Z^1 Y^2 - Z^2 Y^1}$, & $y = \dfrac{Z^1 A^2 - Z^2 A^1}{Z^1 Y^2 - Z^2 Y^1}$. [Note that the denominators will always be the same,

and that the method used to solve for z and y will be discussed below.] If three equations and

three unknowns z, y, & x, we find [similarly]:

An examination of these formulas provides this general rule [The formula is

assumed to have been followed from the equations where $A^1, A^2, A^3, A^4, \& c.$ are equal to a

given polynomial, to the rearrangement through his algorithm leading to the equations above

where the variables x, y, & z have been solved for]. The number of equations & of

unknowns being n, we find the value of each unknown by forming n fractions with a

common denominator as many terms as are various arrangements of n different things

[So, two equations with two unknowns lead to: $2 \times 1 = 2\ terms$ in the numerator and

the denominator alike, three equations with three unknowns lead to: $3 \times 2 \times 1 =$

$6\ terms$, and so on]. Each term is composed of letters $ZYXV\ \&\ c.$ always written in the

same order, but which are distributed, as exponents, the first n numbers arranged in all

possible ways. Thus, when we have three unknowns, the denominator has [1 x 2 x 3 =]

6 terms, composed of three letters ZYX, which in turn receive successively the

exponents 123, 132, 213, 231, 312, 321 [i.e. the 6 terms composed of ZYX combined with

the list of exponents in the given order are:

$Z^1Y^2X^3, Z^1Y^3X^2, Z^2Y^1X^3, Z^2Y^3X^1, Z^3Y^1X^2, Z^3Y^2X^1$]. Given in following terms signs + or –,

according to the following rule. When an exponent is immediately followed in the same

term by an exponent smaller than it, I call that a *derangement*. That taken to account

for each term, if it [the number of derangements] is even or zero, the term has the sign +;

if it [the number of derangements] is odd, the term has the sign –. By example, given the

term $Z^1Y^2V^3$ there is no derangement; this term will therefore have the sign +. The

term $Z^3Y^1X^2$ has the sign + because it has two derangements, 3 before 1 & 3 before 2.

But the term $Z^3Y^2X^1$, which has three derangements, 3 before 2, 3 before 1, & 2 before

1, has the sign – .

The common denominator being thus formed, will have the value of z in the

denominator giving the numerator that which is formed by changing all its

[denominator] terms, [from] Z to A. And the value of y is the fraction that has the same

denominator & of the numerator the quantity that results when changing [from] Y to A,

all terms in the denominator. And [there is] found in similar manner the value of the

other unknowns.

Generally speaking, the problem is determined. But there may be particular

cases where it is undetermined & others where it becomes impossible. This is when the

common denominator is equal to zero; that is to say, if there are only two equations,

when [they have the form]:

$$Z'Y^2 - Z^2Y' = 0,$$

if there are three [equations], when [they have the form]:

$$Z'Y^2X^3 - Z'Y^3X^2 - Z^2Y'X^3 + Z^2Y^3X' + Z^3Y'X^2 - Z^3Y^2X' = 0, \&c.$$

So, if the quantities $A^1, A^2, A^3, \&c.$ are such that their numerators are also equal to zero,

the problem is undetermined; because the fraction $\frac{0}{0}$, which ought to give the value of

the unknowns, is undetermined. But if the quantities $A^1, A^2, A^3, \&c.$ are such that the

common denominator is zero, the numerator or some of [the values] among them are

not zero, the problem is impossible, or at least the unknown variables that can solve all,

or part, are infinite.

For example if you have the two equations $2 = 3z - 2y$ & $5 = 6z - 4y$, from

which we find $z = \frac{2}{0}$ & $y = \frac{3}{0}$. Therefore z & y are infinite variables, which are to each

other the ratio of 2 to 3. By identifying unknowns by ordinary methods, it would fall

into the absurd equation $\frac{2}{3}=\frac{5}{6}$. For the first equation gives $z=\frac{2}{3}y+\frac{2}{3}$ & the second

$z=\frac{4}{6}y+\frac{5}{6}$. Therefore $\frac{2}{3}y+\frac{2}{3}=\frac{4}{6}y+\frac{5}{6}$ or $\frac{2}{3}=\frac{5}{6}$ which is absurd, if z & y are finite

quantities. But if they are infinite, we can say without absurdity that $z=\frac{2}{3}y+\frac{2}{3}$ & at

the same time $z=\frac{2}{3}y+\frac{5}{6}$; because the finite quantities $\frac{2}{3}$ & $\frac{5}{6}$ being nothing compared

to the infinite variables z & $\frac{2}{3}y$, the two equations $z=\frac{2}{3}y+\frac{2}{3}$ & $z=\frac{2}{3}y+\frac{5}{6}$ both are

reduced to $z=\frac{2}{3}y$, which is nothing contradictory (2, 656-659). ■

As previously stated, Cramer's method for solving a system of linear equations is fairly

straightforward, but unfortunately he never included a proof of his work. There are a variety of

ways to prove Cramer's explanation however. Colin MacLaurin, who worked with Cramer,

provided a proof for solving square linear systems in the cases of two and three unknowns, and

briefly mentions the case of four unknowns. This proof was provided in MacLaurin's book *A

Treatise of Algebra* which is also available in the Linda Hall Library rare book room and

incidentally was published in 1748, two years prior to Cramer's book. The following proof is

from Chapter XII of his book and is transposed to modern English by me.

CHAP XII.

Containing some General Theorems for the exterminating

[of] unknown Quantities in given Equations.

Gabriel Cramer: Over 260 years of crushing the unknowns

<div align="center">THEOREM I.</div>

₰₰86. Suppose that two Equations are given, involving two unknown Quantities,

as,

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

then [it] shall [be proved that:] $y = \dfrac{af - dc}{ae - db}$,

where the Numerator is the Difference of the Products of the opposite Coefficients

["opposite" meaning the coefficients are taken from opposite sides of different equations] in

the Orders [terms] in which y is not found [i.e. they are not the coefficients of y], and the

Denominator is the Difference of the Products of the opposite Coefficients taken from

the Orders [terms] that involve the two unknown Quantities [i.e. the denominator is the

determinant of the 2x2 coefficient matrix formed from the equations in the system above where

the determinant of a 2x2 matrix is: $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$].

[Proof:] For, from the first Equation, it is plain that

$$ax = c - by \ .. \ \text{ and } x = \frac{c - by}{a}$$

from the second, $dx = f - ey$ and $x = \dfrac{f - ey}{d}$

therefore $\dfrac{c - by}{a} = \dfrac{f - ey}{d}$, and $cd - dby = af - aey$

Gabriel Cramer: Over 260 years of crushing the unknowns

$$\text{whence } aey - dby = af - cd, \text{ and } y = \frac{af - cd}{ae - db} .$$

$$\text{After the same Manner [we can find that], } x = \frac{ce - bf}{ae - db} .$$

EXAMPLE I.

$$\text{Suppose } \begin{cases} 5x + 7y = 100 \\ 3x + 8y = 80 \end{cases}$$

$$\text{then } y = \frac{5 \times 80 - 3 \times 100}{5 \times 8 - 3 \times 7} = \frac{100}{19} = 5\frac{5}{19} \quad \text{and} \quad x = \frac{240}{19} = 12\frac{12}{19}$$

EXAMPLE II.

$$\begin{cases} 4x + 8y = 90 \\ 3x - 2y = 160 \end{cases}$$

$$y = \frac{4 \times 160 - 3 \times 90}{4 \times (-2) - 3 \times 8} = \frac{640 - 270}{(-8) - 24} = \frac{370}{-32} = -11\frac{9}{16} .$$

THEOREM II.

⨏87. Suppose now that there are three unknown Quantities and three

Equations, then call the unknown Quantities x, y, and z.

Thus,

$$\begin{cases} ax + by + cz = m \\ dx + ey + fz = n \\ gx + hy + kz = p \end{cases}$$

Then [it] shall [be proved that:] $z = \dfrac{aep - ahn + dhm - dbp + gbn - gem}{aek - ahf + dhc - dbk + gbf - gec}$,

where the Numerator consists of all the different Products that can be made of three

opposite ["opposite" means taken from opposite sides of different equations again for this

example just as it did in the previous one] Coefficients taken from the Orders in which z is

not found; and the Denominator consists of all the Products that can be made of the

three opposite Coefficients taken from the Orders that involve the three unknown

Quantities [the denominator is equal to the determinant of the 3x3 coefficient matrix where

the determinant of a 3x3 matrix is:

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = (aei + bfg + cdh) - (ceg + afh + bdi)].$$

[Proof:] For, from the last, it appears, that

$$y = \frac{an - afz - dm + dcz}{ae - db}, \text{ and that } y = \frac{ap - akz - gm + gcz}{ah - gb}; \text{ therefore}$$

$$\frac{an - afz - dm + dcz}{ae - db} = \frac{ap - akz - gm + gcz}{ah - gb}, \text{ and}$$

$$an - afz - dm + dczxah - gbxan - afz + gbdm - gbdcz$$

$$= ap - gm - akz + gczxae - dbxap - akz + gbdm - gbdcz$$

Take $gbdm - gbdcz$ from both Sides, and divide by a, so shall [the result be:]

$$an - dm - afz + dczxh - gbn + gbfz = ap - gm - akz + gczxe - dbp + dbkz .$$

Transpose and divide [solve for z] so shall you find,

$$z = \frac{aep - ahn + dhm - dbp + gbn - gem}{aek - ahf + dhc - dbk + gbf - gec}$$. The values of x and y are found after the

same Manner, and have the same Denominator. For example

$$y = \frac{afp - akn + dkm - dep + gcn - gfm}{aek - ahf + dhc - dbk + gbf - gec}$$.

If any Term is wanting [missing] in any of the three given Equations, the Values of

z and y will be found more simple [or will be easier to calculate due to multiplication by

zero]. Suppose, for Example, that f and k are equal to nothing [$f=k=0$], then the Term fz

will vanish in the second Equation, and kz in the third, and [we will have:]

$$z = \frac{aep - anh + dhm - dbp + gbn - gem}{dhc - gec} \; ; \; y = \frac{gcn - dep}{dhc - gec}$$.

If four Equations are given, involving four unknown Quantities, their Values may

be found much after the same Manner, by taking all the Products that can be made of

four opposite Coefficients, and always prefixing contrary Signs to those that involve the

Products of two opposite Coefficients (3, 81-85). ∎

What I find incredibly interesting about Cramer's work compared to that of MacLaurin is

that they both use a similar method for solving a system of linear equations, but MacLaurin's

work is easier to follow and he offers a clear proof of how it works. Another interesting fact is

that MacLaurin published his work two years before Cramer, yet today we have Cramer's Rule

and not MacLaurin's Rule. The big difference between the works and probably the reason we

use Cramer's Rule is that Cramer explains how to solve the system for n equations with n

Gabriel Cramer: Over 260 years of crushing the unknowns

unknowns and so he gives the more general rule, whereas MacLaurin offers only two finite

examples and a short description of a third.

In order to tie everything together and give a complete understanding of what is

happening in both examples, I offer a modern example of Cramer's Rule used to solve the linear

system $A\mathbf{x} = \mathbf{b}$. Note that for Cramer's Rule to work, it must be applied to a system that has

the same number of linear equations as unknowns just as Cramer explained in his work, i.e. a

square coefficient matrix, and the determinant of the matrix must not be equal to zero

otherwise Cramer's Rule cannot be applied.

Modern Example:

For any $n \times n$ matrix A and any vector $\mathbf{b}$ that is in $\mathbb{R}^n$ (an n-dimensional vector space

over the field of real numbers), allow $A_i(\mathbf{b})$ to be the matrix that is obtained when the ith

column from A is replaced by the vector $\mathbf{b}$. In solving $A\mathbf{x} = \mathbf{b}$ Cramer's Rule solves for the

vector $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ in the following manner: $x_i = \dfrac{\det A_i(\mathbf{b})}{\det A}$, with $i = 1,2,3,...,n.$

For example, given the linear system:

$\begin{cases} 7x_1 - 6x_2 = 8 \\ -8x_1 + 7x_2 = 5 \end{cases}$, where $\mathbf{b} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$, and the coefficient matrix is $A = \begin{bmatrix} 7 & -6 \\ -8 & 7 \end{bmatrix}$,

then $A_1(\mathbf{b}) = \begin{bmatrix} 8 & -6 \\ 5 & 7 \end{bmatrix}$ and $A_2(\mathbf{b}) = \begin{bmatrix} 7 & 8 \\ -8 & 5 \end{bmatrix}$.

Gabriel Cramer: Over 260 years of crushing the unknowns

By Cramer's rule we have:

$$x_1 = \frac{\det A_1 (\mathbf{b})}{\det A} = \frac{\det \left(\begin{bmatrix} 8 & -6 \\ 5 & 7 \end{bmatrix} \right)}{\det \left(\begin{bmatrix} 7 & -6 \\ -8 & 7 \end{bmatrix} \right)} = \frac{(8 \times 7) - (5 \times -6)}{(7 \times 7) - (-8 \times -6)} = \frac{86}{1} = 86$$

$$x_2 = \frac{\det A_2 (\mathbf{b})}{\det A} = \frac{\det \left(\begin{bmatrix} 7 & 8 \\ -8 & 5 \end{bmatrix} \right)}{\det \left(\begin{bmatrix} 7 & -6 \\ -8 & 7 \end{bmatrix} \right)} = \frac{(7 \times 5) - (-8 \times 8)}{(7 \times 7) - (-8 \times -6)} = \frac{99}{1} = 99$$

Therefore, $x = \begin{bmatrix} 86 \\ 99 \end{bmatrix}$ solves the given system.

The great thing about using Cramer's Rule is that it works for any size $n \times n$ system, exactly as described in Gabriel Cramer's book. The only big difference between Cramer's method and what we use today is that determinants are used and we have a couple of different methods of finding them. Both Cramer's method and the modern one yield the same solution and have the ability to solve the system, provided it has an answer to be found. It is fascinating that a technique developed over 260 years ago is still incredibly relevant and widely used today and very little about it has changed over the years. The great mathematicians that worked with Cramer truly helped to revolutionize the world of mathematics and from their collaboration we have today even more incredible advances thanks to their groundwork.

References

1. "Cramer Gabriel." Complete Dictionary of Scientific Biography. 2008.*Encyclopedia.com.* 21
Apr. 2013 <http://www.encyclopedia.com>. In text citation format: (page, paragraph).

2. Cramer, Gabriel. *Introduction à l'analyse des lignes courbes algébriques*. University of
Lausanne. chez les frères Cramer et C. Philibert, 1750. Pages 656 – 659. Print.

3. MacLaurin, Colin. *A Treatise of Algebra*. University of Edinburgh. London, 1748.
Pages 81 – 85. Print.