Paradigms and Mathematics: A Creative Perspective

Hood College MATH 470 Final Project: Dr. Mayfield

Matthew S. Shives—mss9@hood.edu

20311 Parkwood Court

Hagerstown, MD 21742

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1 Introduction

Pervading society is an almost unanimous conception of science, and particularly mathematics, as the branch of learning to which utter certainty of deduction can uniquely be ascribed. When it is considered at all, scientific progress is seen as essentially linear, consisting of the continual shedding of unscientific misconceptions and the straightforward constructing of a body of mathematical and scientific knowledge about reality [46]. Though unwonted perspicacity may quicken the rate of development, it is plain that it can achieve no basic change in the understanding of science itself. This Platonic interpretation of mathematics as a body of transcendent, incontrovertible truth that is gradually unfolded to mankind by the penetrating intellect of its scholars is the very ideal that first drew many mathematicians to their chosen profession; nevertheless, it was shaken to its core by Thomas Kuhn's *Structure of Scientific Revolutions* (1962). Against all conventional wisdom, Kuhn postulated that the history of science is characterized by a progression of paradigm shifts, in which scientists' very understanding of the nature of science changes [32]. Not only does this Kuhnian theory of scientific revolutions describe such events as the discovery of atoms, but it has also been alleged to apply to mathematical developments such as the process of the legitimization of irrational numbers [21]. Still, is it not worse than misguided to claim revolutionary change in mathematics, which is never invalidated but instead expanded? In light of the division this question causes among the greatest minds in mathematics, either veritably recondite acumen or patent arrogance would be requisite to any attempt to bring resolution to the issue. Nonetheless, an inductive, creative approach holds strong potential to shed fresh light on the relevance of revolutions to mathematics.

2 History of the Debate

In order to comprehend the current dialogue on the pertinence of revolutions to mathematics, it is necessary to attain a broad understanding of the genesis of the debate. The concept of revolutions in the history of knowledge first came into vogue through the work of Thomas Kuhn [8]. Investigating the historiography of earlier historians of science such as Alexandre Koyré and comparing it to that of the current scientific orthodoxy, Kuhn was intrigued to see that while the latter looked to history to find the origins of current knowledge, the former in far more interesting fashion traced the evolution of human understanding of science [32]. Noticing the radical shifts in scientific thought that are apparent in Koyré's history but invisible in the other, Kuhn conceived of a new perspective on the development of science. Against the prevailing presumption that scientific progress occurs gradually as the result of the accumulated contributions of individuals, Thomas Kuhn propounded the notion that such seasons of "normal science" are periodically interrupted by great breakthroughs in human understanding, or "paradigm shifts," that alter the very way a subject is perceived [32]. These revolutions inaugurate a new state of awareness that is incommensurable with the previous way of thought; that is, the two paradigms are not comparable, because the very frame of reference has changed [41]. Each paradigm in a given field, such as geocentrism or Euclidean geometry, is no less scientific and rational than the one that succeeds it [31]. Nevertheless, new insights, cultural factors, and the accumulation of perceived violations of the accepted order lead to a reconceptualization of the entire discipline. Hence, the very model of science itself changes continually.

Rather predictably, the initial reception given to this new historiographical perspective was less than enthusiastic. As is reasonable given the fierce methodological disputes in the social sciences, many in that sphere of study showed interest when *The Structure of Scientific Revolutions* was published; in contrast, the scientific and philosophical response to Kuhn was overwhelmingly hostile [32]. Though he had previously expressed ideas that could be viewed as proto-Kuhnian, the eminent scientific philosopher Karl Popper immediately expressed vehement disapproval of Kuhn's thesis of incommensurability, rejecting the perceived implication that scientific "progress" may simply be reactionary, almost random, movement away from one framework into another and not evolution towards truth [44]. Imre Lakatos concurred, writing that in Kuhn's work changes in scientific thought are attributed to "mob psychology" [40]. *Nature* labeled Kuhn and his colleague Paul Feyerabend "the worst enemies of science," and more purely philosophical works, such as those of Dudley Shapere and Israel Scheffler, deplored the support that Kuhn seemed to lend to epistemological relativism, despite Kuhn's denunciation of this reading of his treatise [41, 40]. Meanwhile, Kuhn's language of "paradigm shifts" rapidly insinuated itself into history, business, and nearly every field other than science, much to his mortification; in Kuhn's view, his contribution had been to incorporate ideas common in those fields into the history of science [40]. Even in the light of these perceived failures, however, *The Structure of Scientific Revolutions* has performed a vital function, provoking into existence the study of the historiography of science [11]. Indeed, despite highly negative initial reactions, and despite persistent universal condemnations of Kuhn's doctrine of incommensurability, his wider theory came to be seen as broadly useful. By means of such works as Bernard Cohen's *Revolution in Science* (1985), Kuhnian paradigm shifts have become a mainstream element of historiographic analysis in science, though the debate is far from over [3]. Still, can it be seriously hypothesized that such revolutions are a significant force in mathematical history? Perhaps a pair of case studies will enable more satisfactory elucidation of this enigma.

3 Deduction in Greece

Even the most cursory study of Greek mathematics, especially in the context of other ancient mathematics, induces a startling realization: while absent in essentially every other contemporary society, the concept of proof was apparently integral to Greek mathematics from the very beginning [38]. What was the nature of this fundamental difference in mathematical methodology? Prior to the advent of classical Greece, in regions such as Egypt, Mesopotamia, and China, mathematics was usually perceived primarily as a computational tool for surveying, accounting, and similar pursuits [19]. Even when mathematics was studied for its own sake, the emphasis was on the result and not on its justification.[1] In contrast, though computational activity still took place in Greece, Greek mathematicians primarily endeavored not to find methods to solve practical problems but rather to derive the truth of mathematical propositions. To the Greeks, deduction was pivotal; the goal of mathematics was the demonstration of truths about geometrical figures and rational numbers, and the correct methodology for the attainment of this goal was proof from first principles [28].

[1] One segment of the Egyptian Rhind papyrus reads as follows (omitting units): "Method of calculating a circular area of [diameter] 9; What is its amount as area? You shall subtract its 1/9 as 1, while the remainder is 8. You shall multiply 8 times 8. It shall result as 64" [27, p. 29].

How did this perception of the methods and ends of mathematics, so different from that in the rest of the world and so indispensable to the rise of Western mathematics, develop in Greece? Unfortunately, the most generous characterization of present historical knowledge of this vital era is that it is dubious, due to the impermanent media on which the Greeks chose to record their discoveries [7]. The earliest influential Greek mathematician of whom there still remains a record is Thales of Miletus (c. 600 BCE), who reputedly brought his knowledge of the subject from Egypt and is commonly credited with the demonstration of five basic geometrical truths, including the fact that a circle is bisected by any diameter [18]. It would seem, then, that the Greek deductive approach and focus on abstraction was present from the very birth of Greek mathematics [6]. Since none of the works of Thales are extant, it is impossible to ascertain the verity of this claim; in fact, many historians of mathematics consider it unlikely that Thales accomplished anything approaching rigorous proof [24]. Nonetheless, it is clear that his successors, including the Pythagoreans, Hippocrates, and Eudoxus, followed as one in his footsteps, continuing the process of examining mathematics with logical rigor and abstracting it from everyday problems [35]. This approach reached its climax in the creation of the *Elements* of Euclid, the earliest extant major Greek mathematical work [17]. Though in China, India, and elsewhere mathematicians would indeed develop concepts of proof and attempt to show the truth of certain mathematical results, no mathematician in another culture ever accomplished the rigorous deductive construction of the whole of known geometrical truth from a set of self-evident axioms, as did Euclid, providing a foundation for mathematical inquiry that would remain useful for thousands of years [6].

While it is possible to overstate the significance of the Greek contribution to mathematics—for instance, not all of Euclid's attempts at proof were strict successes—it is at least equally undesirable to miss the revolutionary nature of Greek mathematics. What was the singularity that provided the impetus for the Greek transformation of mathematics from a computational tool to an abstract, axiomatic, and self-justifying field of study? The simplest answer, and perhaps the most correct, is that by approaching mathematics much as they did philosophy, the Greeks endeavored to subject all results to rigorous proof; this method culminated in the cohesive, axiomatic system unfolded in Euclid's *Elements* [29]. In reality, such a characterization may be somewhat disingenuous. Greek philosophy and mathematics did not usually coexist to a significant degree in one individual. Additionally, much of the development of the deductive foundation of Greek mathematics occurred before the philosophers Plato and Aristotle established syllogistic logic's centrality to philosophy, undermining claims of causality in the opposite direction [43, 35]. On the other hand, the originator of Greek mathematics, Thales of Miletus, also generated an influential philosophical school, and Plato promoted geometry as an integral ingredient of his quadrivium, insisting that it be taught at his Academy and famously inscribing over its door, "Let none ignorant of geometry enter here" [7, 2]. Indeed, as Euclid, Archimedes, and Apollonius all worked well after Plato and Aristotle, significant influence of the latter upon the former is almost certain [38]. For modern scholars so far removed from that era, confident disentanglement of these threads is unattainable. Nevertheless, it is apparent even from this overview that though mutually reinforcing interaction occurred between deduction in mathematics and philosophy later in Greek history, emphasis on rigorous logical analysis in Greek mathematics developed before its counterpart in philosophy [6]. An innovation in philosophy did not spread to mathematics; instead, deduction seems to have developed independently in both, and in a way that was almost invisible to the Greeks themselves [24]. Logic was practiced and abstraction performed because in that culture it was apparently unthinkable that they would not be. Possibly because of the existence of a Greek class structure that supplied extensive leisure for reflection, comparable to that in early Virginia or pre-revolution France, it seems that the Greek elite applied deduction indiscriminately, unaware of the pioneering nature of such analysis [29].

4 Analytic Geometry

Despite the strongly suggestive nature of this example, it is judicious to examine another, the Cartesian genesis of analytic geometry. As has been implied, it would be difficult not to see the Greek transformation of mathematics into an abstract, axiomatic, and proof-based subject as revolutionary, if the word be used in anything like its colloquial sense. In the hands of the

Greeks, humankind's very understanding of the nature of mathematics was transfigured; however, was this instance of shifting paradigms a one–time event, or do such revolutions pervade mathematical history? The case of the genesis of analytic geometry may illuminate the issue. In 1637, René Descartes transfigured mathematical thought with the treatise *La Géométrie*, which he published as an addendum to his famed *Discours de la Méthode* [15, 16]. In *La Géométrie*, Descartes astutely linked the hitherto distinct realms of algebra and geometry, defining curves by means of mathematical formulæ [30]. Previously, properties of shapes such as lengths, angles, and areas had commonly been quantified and related by means of equations; well-known examples include the formula for circular area and the Pythagorean theorem. Leaping far beyond this, Descartes and his early followers conceived of geometrical figures like parabolas as being defined in terms of a coordinate system and generated by equations [19]. Rather than understanding equations as being in terms of fixed unknowns, the new analytic geometry allowed unknowns to be variables, taking on every value in a specified range to generate a graphical figure. Descartes' breakthrough changed the course of mathematical history forever; by applying algebraic analysis to geometry and enabling the visualization of algebraic problems in terms of curves, he laid the foundation necessary for Leibniz and Newton's construction of calculus [39].

Although Descartes' discovery was of a specific mathematical technique, in contrast with the Greek metamathematical contribution, it appears just as plainly revolutionary. However, analytic geometry did not spring fully formed from the mind of Descartes; mathematicians throughout history, including Menaechmus, Apollonius, and Nicole Oresme, had reached the very brink of anticipating Descartes' discovery, sometimes even deriving the equations of curves in pseudo–coordinate systems but never deriving curves from equations [5, 1, 42]. Additionally, Descartes' friend Pierre de Fermat independently developed some of the most important elements of the subject almost simultaneously [39]. In a work that remained unpublished during Fermat's lifetime but was circulated in 1636 within a small community of French mathematicians that included Descartes, just before Descartes' own publication of *La Géométrie* but after Descartes had documented his discovery in a 1628 letter, Fermat introduced a core concept of analytic geometry: "Whenever in a final equation two unknown quantities are found, we have a locus, the extremity of one of these describing a line, straight or curved" [5, p. 75]. Though he failed to formulate the modern concept of Cartesian coordinates, Fermat thus pioneered the study of curves as defined by indeterminate equations in two variables [5]. On the other hand, even though the modern concept of coordinates was implicit in his work and he acknowledged the possibility of using his method to analyze indeterminate equations, Descartes understood the primary utility of the analytic geometry he introduced to be the solving of determinate equations [6]. While Fermat pursued the aims that would come to characterize analytic geometry, he was unable to fully shed traditional methodology; conversely, Descartes developed the methods of analytic geometry much more comprehensively while retaining the ends of previous mathematicians [5]. Hence, neither Descartes nor Fermat was able to perform the synthesis that would give the discipline its modern form. In fact, as Fermat's work remained unpublished and Descartes' was highly cryptic, boasting more than explaining, many years elapsed before the labors of such men as Frans van Schooten and John Wallis established the meaning, importance, and even legitimacy of analytic geometry [28].

Why did analytic geometry develop in this way? Though it undermines the revolutionary nature of the discovery somewhat, it is not surprising that Fermat and Descartes were unable to fully exploit the possibilities inherent in their pioneering method. Indeed, that portion of the narrative indicates a key feature of the larger history of analytic geometry. Whereas the Greek paradigm of mathematical deduction appears to have derived directly from the Greek cultural system and was thus present in essentially complete form from the beginning as an unconscious component of Greek mathematics, the development of analytic geometry was a conscious, non–culturally–induced process and, therefore, was not immediately comprehended in its entirety even by its originators [5]. Initially, it seems much more difficult to account for the timing of the discovery of analytic geometry; why, when all but the final step had been taken by geometers as early as the Greek Menaechmus, did the world of mathematics need to wait for Descartes and Fermat to make the climactic breakthrough? Academic consensus is that a related development, that of modern algebraic notation, accounts for the long-delayed, essentially simultaneous, and independent conception of analytic geometry by two separate mathematicians [28]. Without algebraic notation, it makes little sense to generate figures from equations. The truth of this claim is corroborated by the historical fact that Fermat developed his analytic geometry while studying Apollonius and François Viète, who were respectively the ancient mathematician who came closest to analytic geometry and a near-contemporary popularizer of pseudomodern algebraic notation [5]. Further confirmation is found in the fact that Descartes, who carried the method of analytic geometry even further, did so when he had just completed his other major mathematical accomplishment, the formulation of algebraic notation in almost exactly modern form [24]. Evidently, the discovery of analytic geometry occurred just when algebra had advanced far enough to permit it [6]. Still, another vital factor also seems to have been at work; as individuals to whom mathematics was a hobby, not a professional pursuit, both Fermat and Descartes approached mathematics with a fresh eye, uninhibited by the traditional conceptions of mathematics inculcated by mathematical education. As a result, they were able to transform mathematics in a way that had been hitherto inconceivable [5, 9].

5 Revolutions in Historiography

Can a synthesis of these two cases shed useful light on the applicability of paradigm shifts to mathematics? Both developments are indubitably turning points in the history of mathematics, yet the particular details of the formation of Greek abstraction and analytic geometry differ widely. The Greeks contributed a metamathematical shift, transforming mathematics from applied computation into an abstract, logical system deduced from axioms and studied for its own sake. In far less sweeping fashion, Descartes supplied a revolution in mathematical methodology, intertwining the disciplines of algebra and geometry by defining geometric figures in terms of algebraic formulæ. Greek abstraction was essentially unconscious, occurring as an almost necessary consequence of Greek culture; Descartes' development of analytic geometry was fully deliberate and the result of forces internal to mathematics, though it did stem from interaction between two disparate fields of mathematics. In consequence, the Greek establishment of a deductive paradigm of mathematics was simultaneously unprecedented and sudden, while Descartes' analytic geometry was much prefigured and highly incomplete. Strangely, Greek abstraction was an event that was easily accepted and had no evident individual source, even though it was one that remained largely local to Greek culture for centuries. In reverse manner, analytic geometry faced much initial opposition and stemmed directly from two men, Fermat and Descartes, yet became rapidly global. Apart from the importance of both episodes to mathematics, only one similarity is readily apparent: both shifts created a new framework that almost fully superseded the one that had preceded it. Clearly, the construction of a definition of "revolution" in light of these examples remains difficult. Indeed, considering the disparity of detail between the two cases, it is far from certain that any veritable commonality exists between them; perhaps the construct of a "paradigm shift" truly has little relevance to mathematics. What, then, can be done to remedy this bleak situation?

Yet a third parable, this one of a knotted set of three paradigm shifts, may provide the light necessary to elucidate the issue.[2] The Greeks emphasized deduction in the whole of life just as they did in mathematics; in particular, Aristotle promoted the application of syllogistic logic to all fields of inquiry, especially science [45]. Earning for himself the appellation "The Father of Modern Philosophy," René Descartes revived and expanded this ideology in the 17th century, founding the long–standing Western tradition of rationalism, according to which all truth is attainable through the exercise of human reason and observation [9]. As this philosophy trickled into mathematical institutions, increasing concern emerged over the strict logical security of the foundations of mathematics. In consequence, symbolic logic was developed by George Boole and Gottlob Frege, and multitudes of mathematicians including David Hilbert, Bertrand Russell, and the Bourbaki group attempted to formalize the logical basis of mathematics [26, 39]. In like manner, historians of mathematics and science implicitly understood the purpose of their work to be the chronicling of the essentially linear development of the current conception of science [4]. However, in the 18th century, David Hume, Immanuel Kant, and G. W. F. Hegel began a disturbing and contrary school of thought that would end in the philosophy of men such as Søren Kierkegaard and Friedrich Nietzsche [10]. In short, they challenged the certainty of human perception and questioned that the application of logic could yield a body of certain truth.

[2]Unabashed assumption that these events are indeed revolutions is advantageous for the sake of argument.

By the early 20th century, such thought had infiltrated mathematics [23]. Responding to the inability of Hilbert and Russell to complete their formalization programs, Alfred Tarski and Kurt Gödel proved a series of results that revealed the futility of such attempts, demonstrating the impossibility of proving all true statements in a formal system [48]. Ultimately, in 1962 Thomas Kuhn applied these philosophical and mathematical findings to the historiography of science, arguing that periods of "normal science" are punctuated by "paradigm shifts" or revolutions in which the scientists' perspective on their discipline itself undergoes radical transformation [32]. In somewhat ironic fashion, the very concept of paradigm shifts was thus generated by a series of inextricably intertwined revolutions in human understanding, in which rationalism became qualified by skepticism. Still, how is this relevant to mathematics?

In light of this history, the literature on paradigm shifts in mathematics becomes much more intelligible. As previously discussed, early reactions to Kuhn's *Structure of Scientific Revolutions* were characterized by vehement denial; philosophers warned of the apparent implication of relativism, and the scientific community deplored the seemingly implicit denial of scientific progress [40]. Likewise, mathematicians understandably resented Kuhn's insinuation that progress science in general and mathematics in particular does not necessarily proceed through the straightforward application of a deductive process. After all, for centuries or millennia, mathematicians had made statements like that of Hermann Hankel: "In most sciences one generation tears down what another has built, and what one has established another undoes. In mathematics alone each generation builds a new storey to the old structure" [22, p. 49]. Naturally then, in one of the earliest mathematical responses to Thomas Kuhn, Michael Crowe (1975) postulated that "Revolutions never occur in mathematics" [12, p. 19]. Kuhn, trained as a physicist, had avoided detailed discussion of mathematics in his landmark treatise; hence, it was Crowe's paper that broke the uneasy truce and let loose a flood of controversy over the nature and existence of revolutions in mathematics [36]. Herbert Mehrtens published an extensive and thoughtful response to Crowe's brief thesis in 1976, similarly rejecting the applicability of Kuhn's specific theory of the structure of scientific revolutions but arguing in stark contrast to Crowe that Kuhn's general insight is indeed crucial to the historiography of mathematics [37]. To the present day these two seminal viewpoints have shaped the still–unresolved debate. However, in contrast to Crowe's early visceral rejection of revolutions, later contributions such as Joseph Dauben's (1984) crucial paper have generally come to assume the existence of revolution–like change in mathematics and have instead debated the usefulness of the word "revolution," what it should be taken to mean in mathematics, and how revolutions occur [14]. For instance, the authoritative anthology by Donald Gillies (1992) consists largely of essays regarding not the relevance but rather the meaning and importance of mathematical revolutions [22]. The early critic Michael Crowe himself published a recantation of his previous position in 1988 and has asserted that "the question of whether revolutions occur in mathematics is in substantial measure definitional" [13, 11, p. 316]. Even more significantly, Crowe recognized that as previously discussed, "a revolution is underway in the historiography of mathematics" [11, p. 316]. This truth is made evident by the casual usage of Kuhnian concepts in a broad range of other respected mathematical works, including those by Raymond Wilder, Paul Feyerabend, Imre Lakatos, George Lakoff, and Rafael Núñez; description of the traditional view of linear mathematical progress as naïve "Whig history" has even entered the mainstream of textbooks on the history of mathematics [47, 20, 33, 34, 25].

6 Inductive Insight

Unexpectedly, the above analysis of the roots of Kuhn's historiographical perspective reveals the most productive approach to the question of mathematical revolutions. Taking the historical viewpoint that the analysis suggests and interpreting the mathematical debate over paradigm shifts as part of the turmoil that often accompanies a revolution in any field, both the early misunderstanding of Kuhn's ideas and the disparity of opinion that persists to this day are far more understandable. Indeed, this perception brings a glaring deficiency in the mathematical literature into sharp focus. While scholars engage in protracted and acrimonious discussion over the definition, exact mechanism, and applicability of revolutions to scientific and particularly mathematical history, little use of the concept is actually made in historical analysis. The revolution in historiography remains incomplete; though the old paradigm has been rejected, the new paradigm remains nascent [46]. Hence, history languishes while historiographers dispute. If the purpose of mathematical historiography is to compile an understanding of the nature of mathematical history and how it should be studied, the members of this dialogue should transform it from a debate into a dialectic. Rather than propounding theoretical frameworks of the nature of revolutions in mathematics and endlessly debating which theory best describes the data, creating a false dichotomy between theories that each contain a nugget of truth, mathematical historiographers should examine the historical data and create a concept of revolutions in mathematics that is useful for historical study and explication. As the first step in such a pioneering program, analysis should focus solely on the suggestion of a utilitarian understanding of revolutions for the explanation of mathematical history. Recognizing that the historiographical revolution is not yet complete, and proceeding like Kuhn under the simple assumptions that mathematical truth exists, that it can be discovered, and that scientific progress does indeed occur, serious historians of mathematics should bypass the particulars of the present arcane debate and take an inductive approach to revolutions in mathematics [3]. Although arcane issues such as the question of whether it is in understanding or practice that mathematical revolutions occur will maintain some relevance, they are not central to the proposed approach. Instead, this method will result in a fact-driven, contextualized concept of paradigm shifts, one that enables the application of this potentially fruitful idea of revolutions to history.

How should such a pragmatic understanding of revolutions in mathematics, intended to function as a tool to understand mathematical history, be produced? Extending Thomas Kuhn's rough classification of "normal science" and "paradigm shifts," it is most useful to simply define revolutions to be the unit of non–linear, non–normal change in mathematics, the type of progress unexplained by the traditional cumulative model of mathematical history. Refinement of this broad formulation may then lead to the generation of informative subtypes of mathematical revolutions. For instance, both the Greek assimilation of deduction and the Gödelian rejection of formalism lend themselves to interpretation as revolutions in metamathematics; in both cases, a change in the philosophy of mathematics altered the entire scope of mathematical activity. A purely mathematical category of revolutions might include paradigm shifts like those that took place when irrational numbers and non–Euclidean geometry were introduced. In this class of reconceptualization–type revolutions, a new insight radically alters the understanding of preexisting mathematical knowledge. Yet another type of revolution in mathematics might encompass discoveries where the perception of previous knowledge remained the same, yet the discovered technique largely superseded previous ways of doing mathematics; examples include Descartes' development of analytic geometry and Newton and Leibniz's creation of calculus. Clearly, such categories are neither fully distinct nor all-inclusive, yet they provide fertile ground for further development and particulary for articulate historical discussion. If the history of mathematical progress is properly exposited as a mix of revolutionary change and normal accumulation of knowledge, much can be gained as historians are enabled to discuss both the purely intrinsic and the paradigmatic value of great mathematical contributions.

The definition of a revolution is not all that is important to the concept's applicability to mathematics; it is equally critical that it be understood how and why paradigm shifts do indeed occur. It is tempting to formulate a narrow conception of the mechanism of revolutions and espouse it dogmatically. For the purposes of mathematical history, however, it is far more pragmatic to construct a repertoire of plausible mechanisms whose utility may be compared for the exegesis of historical episodes. For instance, a revolution in mathematics may be sparked by a chance discovery, a perceived contradiction or deficiency in the present paradigm, or by a revised viewpoint imported from the general culture. Examples of such revolutions caused by these factors include, respectively, the invention of calculus, the birth of analytic geometry, and the Greek axiomatization of mathematics. Such explanations are plainly not mutually exclusive; a multiplicity of causes is often evident. The actual historical development of paradigm shifts varies in like manner. Sometimes the insight is sudden and complete, with little prefigurement. Other times, for instance in the case of analytic geometry, the crucial method developed very gradually, and still other times, as in Greece, it is the result of the concerted effort of individuals. It is not sufficient, however, that a discovery be made; for a revolution to take place, a paradigm must become widely accepted. How does this happen? Sometimes, the same societal factors that influenced the shift encourage its acceptance. Often, the new technique is obviously desirable, even from within the perspective of the previous paradigm. Occasionally, no conversion does take place; instead, young mathematicians entering the field are trained in the new paradigm, and the old one thus gradually disappears. When historians come to recognize the significance of all of these factors, and of many more as well, their elucidation of the progression of mathematical history will become much more informative.

7 Conclusion

Just as Descartes and Fermat were able to envision and develop a possibility that no professional mathematician had ever glimpsed, so also this undergraduate paper, though far from competent to compete intellectually with the likes of Karl Popper, Thomas Kuhn, and Imre Lakatos, has uniquely considered a separate issue that evidently has been too large to be fully appreciated by those steeped in the lore of the debate. Philosophers and historiographers, so justifiably fascinated by the debate over the question of paradigm shifts in the history of mathematics, have generated reams of valuable dialogue on the issue but have failed to recollect the end purpose of historiography and accordingly establish relative consensus on what basic conception of revolutions can be useful in the writing of mathematical history—for it is evident that in some manner, both the Greeks and Descartes, among others, instigated a revolutionary change in the understanding of mathematics. Inspired by Michael Crowe's recapitulation of his evolving perspective on the debate in Gillies (1995), by Paul Feyerabend's infamous *Against Method* (1990), and by the failure of the rationalism–fueled formalistic approach to the foundations of mathematics itself, the unorthodox proposal of a heuristic understanding of mathematical revolutions based on induction rather than on more traditional deduction from the hypothesized nature of mathematics itself holds great promise for practical fertility both as an unblazed trail and as an approach prompted directly by the needs of the historical community. As it is apparent that the mechanism of mathematical progress is still uncertain, perhaps such analysis will not only increase humanity's comprehension of mathematical history, but also provide invaluable insight into how mathematical discovery itself can be stimulated.

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