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On the Foundations of X-Ray Computed Tomography in Medicine: A Fundamental Review of the 'Radon transform' and a Tribute to Johann Radon

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Abstract

Objective: To acknowledge the work and life of the Austrian mathematician Johann Radon, motivated by a historical narrative on the development of the computed tomography scanner. **Methods:** Information was obtained from journal articles, textbooks, the Nobel web site, and proceedings from mathematical symposiums. **Results:** The computed tomography scanner changed the paradigm of medical imaging. This was a direct result of collaboration between Godfrey Hounsfield and James Ambrose in the 1970s. However, the theoretical basis of computed tomography had been published by Allan Cormack a decade earlier, and a generalized solution to the problem had been described by Johann Radon in 1917 (i.e., the 'Radon transform'). Subsequently, both Hounsfield and Cormack were recipients of the 1979 Nobel Prize in Physiology or Medicine for their achievements in computed tomography imaging. **Conclusions:** An appreciation of the Radon transform serves as a prerequisite to gain deeper insight into signal processing in computed tomography. Such insight offers opportunities to advance optimization strategies in health physics relative to computed tomography quality assurance protocols. **Advances in Knowledge:** As we close in on the 100 year anniversary of the publication of the Radon transform, a review of the literature reveals that a wide-ranging treatise on Johann Radon is not available. This paper attempts to correct that oversight.

Key Words: Radon transform; Johann Radon; computed tomography; mathematical modeling.

Part 1. Introduction

When Sir Godfrey Hounsfield (an English-born electrical engineer) introduced his medical x-ray computed tomography (CT) scanner (in 1971, developed at the British company, Electric & Musical Industries, Ltd. [1-3]), diagnostic imaging—as a discipline—was liberated from the constraints of single-plane radiography. More importantly, for the first time, x-ray imaging via the multi-plane CT construct, could be used to view organs. Thus, the utilization of CT, enabled radiologists to more accurately evaluate a greater number of diseases and conditions to the betterment of patient treatment planning. Furthermore, it can be said that the advent of CT renewed a sense of discovery in the field of radiology not felt since its birth, some 75 years earlier (shortly following Wilhelm Röentgen's discovery of x-rays in 1895 [4]).

To more fully appreciate the engineering feat of Hounsfield's work we step back in time, where on October 1, 1971 at Atkinson Morley Hospital,[1] a renowned brain surgery center in London, the clinical value of Hounsfield's prototype CT scanner—a dedicated head scanner—was demonstrated [5,6]. On that date, the first CT scan was carried out by Hounsfield and the neuroradiologist Dr. James Ambrose on a middle aged woman with a suspected frontal lobe tumor. The tumor was surgically removed soon thereafter, and the surgeon reported that the mass "looked exactly like the picture" [6]. Figure 1 shows the prototype scanner, a design schematic, and an image from the first scan. Following subsequent brain scans on 10 additional patients, performed by and validated by Hounsfield and Ambrose, non-invasive examinations of the brain by way of CT became a decidedly viable option in the radiology armamentarium [2,5-7]. It is notable, also, that by 1975, technology had advanced to the point where Hounsfield was able to build the first whole-body CT scanner.

It is often the case in science and medicine that two people work on identical problems, each unaware of the other person's efforts or contributions. With respect to the history of CT, this theme occurs twice. First, unknown to Hounsfield, the theoretical basis for CT had been published in two papers nearly a decade earlier (the first paper in 1963 and a follow-up paper in 1964) by a South African-born American physicist, Allan Cormack, who spent nearly an equal amount of time developing the idea [8,9]. Interestingly, Hounsfield and Cormack relied on different mathematical approaches to the problem; however, the underlying concept was the same—recovering data lost to attenuated x-rays. In subsequent years, analogous to the honor bestowed on Röentgen by the Nobel committee, awarding Röentgen with the first Nobel Prize in Physics (in 1901) for the discovery of x-rays, Cormack and Hounsfield were jointly awarded the 1979 Nobel Prize in Physiology or Medicine "for the development of computer assisted tomography [10]."

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[1]Atkinson Morley Hospital was founded in 1896 in London, England. During the Second World War a neurosurgery unit was established which marked the initial step in a course of developments in the hospital's timeline that eventually led to its notoriety as a preeminent brain surgery center. Atkinson Morley Hospital remained open until 2003 when the neurosurgery services were moved to the newly-built Atkinson Morley Wing of St. George's Hospital (founded in 1733, London, England) [5,6].

Notably, although Cormack recognized broader applications for his tomographic solution, that is, he realized both x- and gamma-rays could be used for imaging and hypothesized that protons could be used likewise—the historical underpinnings of the problem were not known to him. On this point, we encounter the second and final occurrence of the aforementioned theme. In Cormack's 1979 lecture at the Nobel banquet [11], he asserts:

> It occurred to me that in order to improve treatment planning one had to know the distribution of the attenuation coefficient of tissues in the body, and that this distribution had to be found by measurements made external to the body. It soon occurred to me that this information would be useful for diagnostic purposes and would constitute a tomogram or series of tomograms, though I did not learn the word "tomogram" for many years. At that time the exponential attenuation of x- and gamma-rays had been known and used for over sixty years with parallel sided homogeneous slabs of material. I assumed that the generalization to inhomogeneous materials had been made in those sixty years, but a search of the pertinent literature did not reveal that it had been done, so I was forced to look at the problem *ab initio*. It was immediately clear that the problem was a mathematical one.... Again, this seemed like a problem which would have been solved before, probably in the 19th Century, but again a literature search and enquiries of mathematicians provided no information about it. Fourteen years would elapse before I learned that Radon had solved this problem in 1917.

We learn that the key mathematical technique attributed to the tomographic solution is the so-called "Radon transform," named after the Austrian mathematician Johann Radon. Loosely speaking, the technique may be thought of under the auspices of the Radon problem, and more definitively, as a subfield of Fourier analysis (which is a subfield of harmonic analysis). Hence, the Radon transform is a mathematical operator used in signal processing to recover data of a known signal (i.e., the x-ray beam), or in mathematical terms, a function, passing through a region.

Upon realizing *ex post facto* that more than half a century earlier Radon tackled a generalized solution, Cormack grew interested in tracing the lineage of the Radon problem. Ultimately, Cormack was successful in this quest, and today, we are the beneficiaries of the information he synthesized, thanks to a talk he gave at a symposium on applied mathematics featuring lectures on CT (held in 1982 by the American Mathematical Society) [12]. This talk encapsulated Cormack's research at the turn of that decade (late 1970s to early 1980s) on Radon's work by means of correspondences with prominent mathematicians and physicists of that era. In this light, it is remarkable given the utility of the Radon transform (especially the familiarity of this mathematical operator to engineers in the medical imaging industry [13]), that as we close in on the centennial anniversary of its publication, a wide-ranging treatise on Johann Radon is not readily available in the narratives devoted to historical references in mathematics, or similarly, throughout the annals of radiology.[2] Thus, it is the aim of this paper to present the first comprehensive essay on Johann Radon by means of a three-tiered composition. First, under the broader heading of harmonic analysis, the lineage of the Radon problem is presented. Next, the life and esteemed career of Johann Radon is explored. Finally, as the capstone to this paper, a treatment of the Radon transform—ascribed to CT modeling—will be offered in plain mathematical language.

Part 2. Narrative on Johann Radon: A Historical Perspective

1. Lineage of the Radon Problem

As we trace the lineage of the Radon problem we are guided by Cormack, and in fact, we find ourselves indebted to his efforts concerning this subject matter. In this context, the reader is referred to Figure 2, which serves as a roadmap to the following time-line of relevant historical events surrounding the problem, as adopted from Cormack [12]:

- The first person known to tackle Radon's problem was the great Dutch physicist Hendrik Lorentz. He found the solution to the three dimensional problem where a function is recovered from its integrals over planes. If $\hat{f}(p, \vec{n})$ is the integral of f over a plane perpendicular to the vector $p\vec{n}$ from the origin,

[2] It is interesting to note that much of Radon's original works, including his work that is the focus of this exposition, have only (relatively) recently become available in the English language, as will become apparent throughout this paper.

and a distance p from the origin, then f at the origin is given by

$$f(0) = -\frac{1}{8\pi^2} \int \left(\frac{\vartheta^2 \hat{f}(p,\vec{n})}{\vartheta p^2} \right)_0 d\omega_n,$$

where $d\omega_n$ is an element of solid angle in the direction of $\vec{n}$. Since the origin may be arbitrarily chosen, the result holds for any point. Interestingly, it is unknown why Lorentz thought of the problem, or what his method of proof was, for we only know of his work through H.B.A. Bockwinkel. That is, in a paper written in 1906 by Bockwinkel on the propagation of light in biaxial crystals [14], he attributed the above equation to Lorentz.

- Following Lorentz came Radon's famous integral[3] in 1917 [15], such that

$$f(r,\phi) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_{-\infty}^{\infty} \left(-\frac{1}{t}\right) \frac{\partial}{\partial l} p(l,\theta) dl d\theta,$$

  where $p(l,\theta)$ is the density integral (or ray sum) measured along the ray inclined θ with respect to a vertical axis and passing within a distance l from the center of the region being scanned. Further, $f(r,\phi)$ is the density at the point with polar coordinates (r,ϕ) in this region, while $t = l - \cos(\theta - \phi)$ is the perpendicular distance between the ray and this point. When ray sums in a given projection are spaced evenly in l, and projections are spaced evenly in θ, a simple reconstruction method can be found directly from the equation by approximating both integrals by sums and approximating the partial derivative by an appropriate first difference [16].
- The problem again surfaces in 1925, involving two physicists Paul Ehrenfest and George Uhlenbeck. In Uhleneck's paper he credits Ehrenfest for drawing his attention to the results of Lorentz and suggesting that he generalize it to n-dimensions, which Uhlenbeck did using Fourier techniques [17]. Once again there was no reason given for solving the problem.
- In 1935, in Stockholm, Cramér and Wold[4] used the Fourier integral approach to better understand marginal distributions of a probability distribution in order to infer the distribution itself [18].

  Implications to x-ray CT in medicine. As noted by Cormack, marginal distributions can be considered as projections or views in terms of CT scanning.
- In 1936, in Leningrad, the Armenian astronomer Viktor Ambartsumian provided an elegant mathematical solution[5] to a problem posed earlier by Eddington. That is, Ambartsumian determined the distribution of the spatial velocities of stars from the distribution of their radial velocities obtained for various regions of the sky. This find was considered of fundamental importance for the kinematics and dynamics of the galaxy [19].

  Implications to x-ray CT in medicine. With respect to Ambartsumian's solution, Cormack states, "This is just Radon's problem in three dimensional velocity space rather than ordinary space, and Ambartisumian gave the solution in two and three dimensions in the same form as Radon." Cormack also points out, "This is the first numerical inversion of the Radon transform and it gives the lie to the often made statement that computed tomography would be impossible without computers." Cormack states that, "Details for the calculation are given in Ambartsumian's paper, and they suggest that even in 1936 computed tomography might have been able to make significant contributions to, say, the diagnosis of tumors in the head." Reportedly, as Ambartsumian told Cormack—he [Ambartsumian] was informed about Radon's results two years after he [Ambartsumian] published his work.
- In 1947, Szarski and Wazewski describe Radon's problem by formulating it in terms of a set of "fonctions cylindrique" (or "cylindrical functions"), and then state the problem consists of finding whether or not this set of functions tends to a solution.

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[3]In 1986, an English translation of Radon's 1917 paper (translated by P. C. Parks) appeared in the journal *IEEE Transactions on Medical Imaging*, volume 5, number 4, pages 170-176, as titled, "On the Determination of Functions from Their Integral Values Along Certain Manifolds."

[4]Although Cormack originally cited the paper published by Cramér and Wold as the year 1936, search results in the Electronic Research Archive for Mathematics, maintained by the European Mathematical Society, show an alternative publication of the same content one year earlier [18], that is to say 1935.

[5]V. A. Ambarzumian, Mon. Notic. Roy. Astron. Soc. 96, 172 (1936).

- Finally, in 1956, the electrical engineer and radio-astronomer Ronald Bracewell worked out Radon's problem using Fourier and other methods to determine radio emissions from the Sun.

The reader should take note that in the above account of the Radon problem, several references were made to Fourier methods. In a latter section of this essay (A Derivation of the Radon Transform), further references to Fourier techniques will be made. Suffice it to say at this time, that in the bigger picture, such techniques—including the Radon transform itself—fall under the heading of Fourier analysis. Moreover, Fourier analysis falls under the umbrella of harmonic analysis. The three main subclassifications of harmonic analysis, according the American Mathematical Society, are observed in Table 1. The reader should note that the Radon problem, which relates to reconstructing a function from its integral, falls under the second arm, "Harmonic Analysis in Several Variables," and within this subclassification, it belongs under "Fourier and Fourier-Steiltjes transforms and other transforms of Fourier type."

To transition to the next section of this essay on the life and work of Johann Radon, the two areas of mathematics that Radon was most interested in, the calculus of variations and functional analysis, are briefly introduced.

Calculus of Variations: Whereas finding the minimum and maximum of functions is a basic idea in calculus, in the calculus of variations this idea is expanded to finding the extremas of mathematical concepts called functionals [20]. Examples include 1) the length of a line of a curve joining two given points; 2) the area of a surface; 3) moments of inertia of a curve or a surface with respect to a point or an axis or a plane; and 4) the resistance encountered by a physical body moving with given velocity through a medium [20]. Such examples have important implications in engineering and physics.

Functional Analysis: The most important role of functional analysis is that of a mathematical language. More specifically, functional analysis became the language of 20th century mathematics (more precisely its part called analysis) and theoretical physics; much of the subject matter under its umbrella deals with the convergence of functions [21]. Functional analysis includes but is not limited to the mathematics of set theory, topology, measure theory, and linear spaces [22].

2. The Life and Work of Johann Radon

Johann Karl August Radon (December 16, 1887 - May 25, 1956) was an Austrian mathematician who focused his career on the study of analysis. To this end, his dissertation explored aspects of the calculus of variations, and his collective works laid the foundations of functional analysis [23]. Radon was born in Tetschen (near Bohemia, in present-day Czech Republic, then part of the Austro-Hungarian monarchy). He was the only son of Anton Radon and Anna Schmiedekrecht (Anton's second wife). At preparatory school in Leitmeritz (in present-day Czech Republic), Radon enjoyed Latin and classical Greek as well as botany, history, and music; however, his interest in mathematics proved to be the strongest [24]. Following preparatory school (where he graduated with distinction), Radon began his study of higher mathematics at the University of Vienna.

In Vienna, Radon studied under Gustav von Escherich (June 1, 1849 - January 28, 1935). Notably, on par with convictions to promote the development of mathematics in Austria,[6] Escherich had a reputation of impressing upon his students, the works on analysis by Karl Weierstrass [23]. Indeed, today, we recognize Karl Weierstrass (October 31, 1815 - February 19, 1897) as the "Father of Modern Analysis" [25]. Moreover, Esherich, like Weierstrass before him, embraced a rigorous definition of the calculus,[7] much like that proposed by Augustin-Louis Cauchy (an early 19th century mathematical intellect). Hence, it is in this age, regarded by historians as the dawn of the most prolific period in mathematics—in terms of extending our knowledge of and interactions with the universe, one easily finds the inspiration that 20th century mathematicians drew upon. Moreover, it was the knowledge which emerged from and transcended this period that significantly influenced Radon's decisions to pursue his vocation in the field of analysis. The following chronology provides a synopsis of Radon's career.

[6]Gustav von Escherich and Emil Weyr founded the journal *Monatshefte für Mathematik und Physik* in 1890, which was published until 1944. In addition, Gustav von Esherich, together with Ludwig Boltzmann and Emil Müller, founded the Mathematical Society in Vienna in 1903, later renamed the Austrian Mathematical Society (1948).

[7]Although calculus was established over 100 years earlier by Gottfried Leibniz and Isaac Newton (ca. 17th-18th centuries), the credit for our current understanding of this field (that is, calculus more rigorously defined) is most cited with the methods developed by the mathematician, Augustin-Louis Cauchy (August 21, 1789 - May 23, 1857).

1910: Dissertation[8]—Über das Minimum des Integrals $\int_{S_0}^{S_1} F(x, y, \theta, \kappa)ds$, University of Vienna.

1910-1911: University of Göttingen.

1912-1919: Technical University of Vienna.[9]

1913 Habilitationsschrift—Theorie und Anwendungen der absolut additiven Mengenfunktionen,[10] University of Vienna.

- During this period, Radon was also a Privatdozent at the University of Vienna.

1917 The paper on what became known as the Radon transform was published in the journal, *Berichte der Sächsischen Akadamie der Wissenschaft.*

1919-1922: Appointed as Extraordinary Professor at the University of Hamburg.

1922 While at Hamburg, the paper that identified Radon's theorem was published in *Mathematische Annalen.*[11]

– **Radon's theorem.** Any set of $n + 2$ points in $\mathbb{R}^n$ can always be partitioned in two subsets V_1 and V_2 such that the convex hulls of V_1 and V_2 intersect.

1922-1925: Held title of Ordinary Professor, University of Greifswald.

1925-1928: Held title of Ordinary Professor, University of Erlangen.

1928-1945: Held title of Ordinary Professor, University of Breslau.

1945-1947: A period of time interrupted by World War II. Radon traveled to Innsbruck, Austria to escape a siege of Breslau, Poland.

1947:

- Returned to the University of Vienna and held the title of Ordinary Professor.
- Founded the journal *Monatshefte für Mathematik*, which began publication in 1948. (Prior to World War II, this journal had been known as *Monatshefte für Mathematik und Physik.*)

1948-1950: Served as president of the Austrian Mathematical Society (known as the Mathematical Society in Vienna prior to the Second World War).

1951-1952: Dean of the Philosophical Faculty at the University of Vienna.

1954-1956: Rector of the University of Vienna.

It is interesting to point out that even though Cormack had not been aware of Radon's theory of integration, in some academic circles we find that Radon's work in this area (as well as measure theory) was considered classical during his lifetime [23]. Leopold Schmetterer, a colleague of Radon's, recounts that in the 1950s a young American mathematics student came to the University of Vienna to visit him [Schmetterer] for one semester—the student had been a pupil of Antoni Zygmund (a harmonic analyst famous for his work in trigonometric series). Schmetterer recalls that when the student saw the name Johann Radon in large letters on the door of the office next to his, the student asked Schmetterer, "Who is that?" Schmetterer answered saying, "You certainly know the inventor of the 'Radon integral'." The student replied, "Of course, I know him, but he must be at least 100 years old, since these results have long been an essential constituent of measure theory and theory of integration." This story ends in a whimsical matter in that, just then, Radon apparently returned from a lecture and the young American student could finally convince himself that Radon was still alive and scientifically active [23]. In fact, in 1954, two years prior to his death, Radon published an article on the calculus of variations in *Archiv der Mathematik.*[12]

In another recount, Schmetterer tells of a time in which Radon in 1948 read a rather suspicious letter he had received in which its author claimed to have found the "correct value" for π [23]. The letter's author was, however, unable to convince the world of this find, and was asking for the help of Radon's authority to convince them so. The author went on to propose that Radon should deposit the new value with the United Nations, and ask 1 U.S. dollar for each request. In return, the letter's author would willingly share his anticipated immense profit with Radon.

[8]English translation of Radon's dissertation—On the Minimum of the Integral $\int_{S_0}^{S_1} F(x, y, \theta, \kappa)ds$.

[9]The Technical University of Vienna was founded in 1815 as the Imperial-Royal Polytechnic Institute.

[10]English translation of Radon's habilitationsschrift (or post-doctorate thesis), "Theory and Application of Absolute Additive Weighting Functions."

[11]The article in which Radon's theorem appeared was titled, in German, "Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten." Translated into English, this becomes, "Volumes of Convex Bodies that Contain a Common Point."

[12]Radon, J. Gleichgewicht und Stabilität gespannter Netze. (German) Arch. Math. (Basel) 5, (1954). 309-316.

We are fortunate to be given a glimpse into the personal story of Johann Radon through the recollections of his daughter, Brigitte Burkovics.[13] From her unique perspective [24] we see that Radon not only enjoyed [chamber] music and hiking or that he moved frequently in his teaching positions, but like so many other individuals and families in those times, encountered hardships during the First and Second World Wars (including the loss of his youngest son in the Second World War).

In January 1945, Radon moved his family from Breslau, Poland (where he was teaching at the University of Breslau) to Innsbruck, Austria (where he served as a guest to the University of Innsbruck). The move from Breslau was a decision made to escape a siege of Breslau that was soon to begin, and the town of Innsbruck was sought out because one of the sisters of his wife lived there. Above all, however, Radon remained positive [24]. For instance, after having experienced significant loss during World War II, his daughter explains:

> The circumstances of our life had completely changed. The loss of my beloved brothers was very hard for all of us. [Note: all three of Radon's sons died early in life.] Then we had lost our home and all our belongings, we had only saved our lives, and the future was very uncertain. Yet I have never heard my father complaining, neither at this time nor at any other. Only once he mentioned the loss of his very valuable library, which he would have much needed. Though we were always hungry and had no good shoes, we went sometimes hiking in the mountains. The wonderful surroundings of Innsbruck and the hope that the future could only become better, helped us to get through these months. In autumn 1945, the French took over as occupying power from the American forces and in a short time they opened the theatre and the university. Father could begin again as a guest professor. Though the winter was very cold, and many rooms in the university had broken windows, the glass being replaced by paper, we were all very happy that we could go on studying without working besides 8 hours per day for the war industry.

For more details on Radon's personal triumphs and struggles, the reader is referred to the transcript of his daughter's 1992 speech celebrating Radon's life and commemorating 75 years of the Radon transform [24].

Toward the end of his career Radon acquired many administrative responsibilities at the University of Vienna; however, to say that he had a love of paper work in this capacity may be an overstatement. In another somewhat whimsical story, Radon asked Schmetterer to his office to find a form in a stack of papers on his desk for the Ministry of Education [23]. The form dealt with the heating of the rooms of the Mathematical Institute. When Schmetterer began at the top of the pile, Radon remarked, "The relevant geologic stratum must be much further down."

On May 25, 1956, at the age of 68 years, Johann Radon died after five months of illness [24]. An obituary appeared in *Monatshefte für Mathematik* in 1958 [26]; whereas it was written in German, this author is not aware of any English translation of the text. Hence, it is explicitly stated here, and perhaps for the first time—that history remembers Johann Radon for having played a key role in helping rebuild the Austrian mathematical scene following the Second World War. Not only did Radon re-establish the Austrian Mathematical Society as well as the journal founded by his advisor (Gustav von Escherich), but prior to returning to the University of Vienna in 1947 he steadfastly served the Austrian community (during very tumultuous times), particularly during a period of time when he was a guest professor at the University of Innsbruck.

Radon's mathematical legacy is not forgotten. In 2003, the Austrian Academy of Sciences opened the Johann Radon Institute for Computational and Applied Mathematics, which promotes the role of mathematics in science, industry and society [27]. In addition, according to the Mathematics Genealogy Project [28], Radon had 18 students who earned their PhD's, and from these, "312 descendants." Finally, motivated in part by the previously discussed efforts of Allan Cormack to uncover the lineage of the Radon problem (that is, to reconstitute a function from its integrals), the appendix of this paper offers a compiled bibliography of Radon's work.

3. A Derivation of the Radon Transform

In some ways, this section of the essay picks up where a 1996 article published by Friedland and Thurber in the *American Journal of Roentgenology* leaves off [29]. For example, that article not only provided a

[13]Brigitte Burkovics (maiden name: Radon) earned her PhD in mathematics at the University of Innsbruck in 1948 with a dissertation titled, "Series Expansions of the Elliptic Integrals."

succinct history of the theoretical work by Cormack as well as the engineering and clinical works performed by Hounsfield and Ambrose, but it also provided a concise history of the French mathematician Jean Baptiste Joseph Fourier (March 21, 1768 - May 16, 1830), the namesake given to Fourier analysis. (Note: as mentioned previously, the Radon transform falls under a categorical subfield of mathematics known as Fourier analysis.) In addition, like Friedman and Thurber, this author recognizes that almost all CT scanners today employ fast Fourier transform algorithms by means of filtered (convolutional) back projection. Interestingly, it is from this application of the fast Fourier transform algorithm (an efficient computational implementation of the discrete Fourier transform), that images in CT can be rapidly processed using a digital computer [29].

Moreover, it is important to understand that the Radon transform refers to a special case of the Fourier transform; and the Fourier transform is a limiting case of the Fourier series [30,31]. This means whereas a Fourier series is the mathematical instrument used when evaluating periodic phenomena [30], a Fourier transform is reserved for the study of phenomena that is nonperiodic [31]. Thus, the choice of the application of a "transform" is an intuitively simple decision, given that x-ray photons in the exit beam strike the image receptor in burst-like impulses that are mostly nonperiodic rather than periodic in fashion. In mathematical terms, burst-like physical phenomena that are almost periodic are known as line impulses. The concept of the line impulse will be a key point expanded upon below. To simplify the derivation of the Radon transform, assumptions are made that ignore certain computational issues, as follows:

- The playoff between Cartesian and polar coordinate representations, i.e., the 2-dimensional xy-plane versus spherical/circular symmetry, respectively.
- Adjustments in modeling to account for fan-beam [16] or cone-beam CT constructs [32].

To this end, it is a parallel beam configuration in CT that will be described by the model (and thus most enthusiastically applied to first and second generation CT scanners, in line with the historical nature of this paper).

It is noted that no single mathematical derivation exists for x-ray CT in medicine due to the lack of a truly rigorous justification of a tomographic algorithm [33]. Hence, inversion of the Radon transform is described, here, in simple mathematical language.[14] Such interpretation will be facilitated by a glossary of terms, see Table 2. Also, where noted, Wolfram *Mathematica* (the online computational engine, $Wolfram|Alpha^{TM}$) was used to plot the traditional representative line equations of the x-ray photons. Accordingly, then, the steps necessary to invert the Radon transform, without reference to discrete numerical analysis, as it is this inversion technique that serves to recapture the information lost to attenuated x-ray photons, will constitute the balance of this section.

3.1. **The Set Up.** The underlying theme in this mathematical application is a signal processing challenge, and the set up for the analysis is straightforward. We have a 2-dimensional slice of a region of variable density (the patient), and the goal as applied to CT scanning is to reconstruct the resulting x-ray signal (the image) after repeatedly passing x-rays through the region at different angles of initial projection (the CT gantry). More concisely stated, we are measuring the resultant signal at different trajectory lines by accumulating (integrating) the signal after projecting x-ray photons through the region. Hence, the approach reconstructs the densities of the materials interacting with the x-ray photons [34], to ultimately assign density values according to the Hounsfield unit scale of CT numbers for data acquisition/image processing [35]. Such modeling serves as an engineering template for trouble-shooting in the event of errors, such as equipment failure or computer algorithm failures, which may lead to radiation overdose of the patient [36].

Given that the approach resolves signal processing by means of calculating line integrals to recover the intensity of the x-ray signal (i.e., capture the data lost to attenuated or scattered x-rays), a comparison may be made to the inverse square law which estimates beam intensity from known initial conditions, the intensity of- and distance from- the beam [37]. However, the comparison is rudimentary at best because the central and interesting feature of the model applicable here, i.e., the Radon transform and its inverse, lies in the fact that we are strictly calculating the intensity of the exit/secondary beam based solely on a known intensity of the primary beam.

The derivation of the mathematical model can be relatively easy to follow since the steps involved are pragmatic to imaging tasks carried out in the CT suite. We begin by a detailed inspection of representative

[14] The derivation presented is adapted from course notes on EE261 The Fourier Transform and Its Applications as taught by Brad Osgood, PhD, Stanford University Engineering.

x-ray trajectories relative to the CT gantry (i.e., the family of parallel lines), and then compare the suitability of two different proposed coordinate systems for the model.

3.1.1. *Lines/Family of Lines.* Refer to Figure 3 for a depiction of the CT gantry with the x-ray beam drawn as a family of parallel lines though the region. Each representative x-ray trajectory (i.e., the parallel lines) can be written in the slope-intercept form of a line.

$$y = mx + b, \text{ with } -\infty < b < \infty \text{ and } 0 \leqslant \infty.$$

In this form, the coordinates of the lines in the xy-plane are the points (m, b), "m" the slope of the line and "b" the y-intercept. However, this coordinate system breaks down as "m" and "b" vary because the formula is not valid for vertical lines, such that a vertical slope is not defined [38]. Therefore, a more suitable coordinate system is required to parameterize a line (and all families of parallel lines), and therefore, it is interesting to look at what a family of parallel lines may have in common (see Figure 4).

Referring to Figure 4, one such identified commonality is that each line has the same angle to the horizontal axis, the x_1-axis. Thus, we will call this angle, the angle (φ). Specifically, it is the normal vectors of these lines that have the same angle to the x_1-axis. However, to better identify locations of lines, we need more than just the angle to the x_1-axis. To single-out a line we look at its distance (ρ) from the line passing through the origin (see Figure 4). Thus, with these parameters, the distance (ρ) and the angle (φ), we have successfully established an unambiguous coordinate system that is not flawed by the non-existence issue of a vertical slope. The *Cartesian equation of the line for the model*, is now specified by a given coordinate pair (ρ, φ) in the form:

$$x \cdot n = x_1 \cos\varphi + x_2 \sin\varphi = \rho$$

where both x and n are vectors, each defined in the following way, $x = \langle x_1, x_2\rangle$ and $n = \langle \cos(\varphi), \sin(\varphi)\rangle$, and the line equation is derived by vector multiplication, in this case by using the dot product method, where it is said that x is dotted with n.

3.1.2. *Line Impulse.* It is necessary to account for the nonperiodic nature of the signal concentrated along each trajectory taken by the x-ray photons, and this is accomplished by considering the line impulse [31]. The line impulse describes the physical phenomena of x-ray photons striking the image receptor in the CT gantry. To define the line impulse mathematically, we first need to set the *Cartesian equation of the line for the model* to zero as shown.

$$\begin{aligned} \rho &= x_1 \cos\varphi + x_2 \sin\varphi \\ \rho - x_1 \cos\varphi - x_2 \sin\varphi &= 0 \end{aligned}$$

The resultant equation, which immediately follows, then becomes a function of delta, denoted by δ [31],

$$\rho - x_1 \cos\varphi - x_2 \sin\varphi \xrightarrow{\text{becomes}} \delta(\rho - x_1 \cos\varphi - x_2 \sin\varphi).$$

The delta function δ, is the classical approach to the line impulse [30], and has advantageous implications for dimensionality and integration of a line in the following way.

$$\int_L \mu = \iint_{R^2} \mu(x_1, x_2)\, \underbrace{\delta(\rho - x_1 \cos\varphi - x_2 \sin\varphi)}_{\text{line impulse}}\, dx_1 dx_2 \tag{1}$$

On the left hand side of Equation 1, the line integral denoted by L of the function μ is a single integral for the 1-dimensional case of the line, and on the right hand side, the double integral denoted by R^2 of the function μ is the 2-dimensional case of the plane. Note that the line impulse, i.e., the delta function δ concentrated on a line, has a domain of infinity on the line and zero off the line.

3.2. **The Radon Transform.** Equipped with a suitable coordinate system and having addressed the line integral with respect to the line impulse, we are ready to introduce the computational steps central to the mathematical model, inverting the Radon transform. As we do this, it is important to first point out what is varying as we work through the computations, i.e., to identify the variables associated with the integrand (those terms being integrated).

As shown in Figure 5, superimposition of the useful/suitable coordinate system (as described earlier) onto a representative cross-sectional image (the region of interest) will help identify the variable. Looking at Figure 5, think in terms of what it means to fix φ and let ρ vary. This means the family of parallel lines will be defined by the angle made with the x_1-axis, and only the distance of a line from that of the line going through the origin will be of concern. In other words, the angle is fixed, it does not change, allowing ρ to be the variable as we accumulate (integrate) data. Thus, the Radon transform $\mathcal{R}$ can now be introduced by rewriting Equation 1, in relation to the signal/function μ, where μ is a function of ρ and φ, (see Equation 2). *Note: in this and the remaining sections, the x-ray signal will be written as the function* μ.

$$\mathcal{R}\mu(\rho,\varphi) = \int\limits_{L(\rho,\varphi)} \mu = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \mu(x_1,x_2)\delta(\rho - x_1\cos\varphi - x_2\sin\varphi)dx_1dx_2 \tag{2}$$

Accordingly, both the line integral L and the double integral above are more concisely expressed here than in Equation 1. With respect to the line integral, it is now written as a function of ρ and φ, and the limits of integration ($-\infty$ to ∞) are explicitly stated for the double integral.

As discussed earlier the Radon transform is a special case of the Fourier transform, thus it is accurate to write the Fourier transform $\mathfrak{F}$ with respect to ρ (denoted by the subscript ρ) as a function of the Radon transform $\mathcal{R}$, as seen in the following notation [31].

$$\mathfrak{F}_\rho\big(\mathcal{R}\mu(\rho,\varphi)\big) = \int\limits_{-\infty}^{\infty} e^{-2\pi i r\rho}\big(\mathcal{R}\mu(\rho,\varphi)\big)d\rho \tag{3}$$

The significance of this step is that we are now accounting for the spatial domain, denoted here by the letter "r" in the complex exponential, $e^{-2\pi i r\rho}$ [30]. In reality, the derivation for this mathematical application (as we strive to understand it in the context of CT) is concerned with two domains, the spatial domain and the frequency domain, and moreover, both are present/available in the complex exponential, $e^{-2\pi i r\rho}$.

Further evaluating Equation 3, including switching the order of integration (see below), we are able to address dimensionality in the Radon problem (by first dealing with the 1-dimensional component).

$$\begin{aligned}
\mathfrak{F}_\rho\big(\mathcal{R}\mu(\rho,\varphi)\big) &= \int\limits_{-\infty}^{\infty} e^{-2\pi i r\rho}\big(\mathcal{R}\mu(\rho,\varphi)\big)d\rho \\
&= \int\limits_{-\infty}^{\infty} e^{-2\pi i r\rho}\left(\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \mu(x_1,x_2)\delta(\rho - x_1\cos\varphi - x_2\sin\varphi)dx_1dx_2\right)d\rho \\
&= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \mu(x_1,x_2)\left(\int\limits_{-\infty}^{\infty} e^{-2\pi i r\rho}\delta(\rho - x_1\cos\varphi - x_2\sin\varphi)d\rho\right)dx_1dx_2 \\
&= \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \mu(x_1,x_2)\left(\underbrace{\int\limits_{-\infty}^{\infty} e^{-2\pi i r\rho}\delta(\rho - (x_1\cos\varphi + x_2\sin\varphi))d\rho}_{\text{The 1-dimensional Fourier transform.}}\right)dx_1dx_2
\end{aligned}$$

The above multi-line evaluation yields:

$$\mathfrak{F}_\rho\big(\mathcal{R}\mu(\rho,\varphi)\big) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \mu(x_1,x_2)e^{-2\pi i r(x_1\cos\varphi + x_2\sin\varphi)}dx_1dx_2, \tag{4}$$

where the complex exponential, $e^{-2\pi i r(x_1\cos\varphi + x_2\sin\varphi)}$, may be rewritten after distributing "r", such that we obtain $e^{-2\pi i(x_1 r\cos\varphi + x_2 r\sin\varphi)}$. To finish simplifying the complex exponential, we introduce the concept of dual variables, in that (x_1 is paired with ξ_1) and (x_2 is paired with ξ_2), where ξ_1 and ξ_2 are each constants defined in the following way:

$$\xi_1 = r\cos\varphi \text{ and } \xi_2 = r\sin\varphi.$$

Although each of these equalities above suggest implementation of polar coordinates (the coordinate system employed for spherical/circular symmetry), they are not intended to do so in this derivation. The equalities merely serve as a means to express the complex exponential more simply with dual variables, as follows:

$$e^{-2\pi i(x_1\xi_1+x_2\xi_2)}.$$

It is now important to emphasize what has been derived thus far, and what computational steps remain. We have derived the 1-dimensional integral (that integral involving the line impulse). The remaining computational steps in the Radon problem involve the actual processes to recover the values of the densities μ, i.e., to reconstruct the densities from the region, by inverting the Radon transform as a function of the Fourier transform over the 2-dimensional region.

3.3. **Inverting the Radon Transform.** To invert the Radon transform, we first plug the result of the 1-dimensional integral (as derived above and reemphasized below) back into the original Fourier transform which we set up earlier. This is shown below. We now have the 2-dimensional Fourier transform of μ, i.e., the double integral, set up to integrate first with respect to dx_1 and then with respect to dx_2.

$$\mathfrak{F}_\rho\big(\mathcal{R}\mu(\rho,\varphi)\big) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu(x_1,x_2)e^{-2\pi i(x_1\xi_1+x_2\xi_2)}dx_1dx_2 \tag{5}$$

Hence, to best convey the details of the final computation step, it is of certain benefit to pause in order to recapitulate the entire mathematical derivation up to this point.

- First, a suitable coordinate system (ρ,φ) was found.
- Second, "φ was fixed to let ρ vary," where φ is the angle that each line in the family of parallel lines makes with the x_1-axis, and ρ represents the values of distances of these lines from the line passing through the origin.
- Third, the 1-dimensional Fourier transform of the corresponding Radon transform was found with respect to ρ, resulting in the 2-dimensional Fourier transform of μ.
  (a) In principle the problem is solved. We have measured the Radon transform, i.e., the line integral of μ along the family of parallel lines.
  (b) Because we know the expression of the 1-dimensional transform and the values which emerge, those associated with $\left(e^{-2\pi i(x_1\xi_1+x_2\xi_2)}\right)$, we can now compute the Fourier transform with respect to ρ [31].

By computing the Fourier transform with respect to ρ, we get the 2-dimensional Fourier transform with respect to μ. This means that we can find μ by taking the inverse of the 2-dimensional Fourier transform of what was found:

$$\mathfrak{F}\mu(\xi_1,\xi_2) = \mathbb{G}(\xi_1,\xi_2) \xrightarrow{\text{recovers }\ \mu} \mu = \mathfrak{F}^{-1}\mathbb{G}(\xi_1,\xi_2) \tag{6}$$

where $\mathbb{G}(\xi_1,\xi_2)$ equals $\left(e^{-2\pi i(x_1\xi_1+x_2\xi_2)}\right)$, the known values of the 1-dimensional Fourier transform. By taking the inverse of the signal/function μ (we recover the lost data contained in the trajectory lines of the x-ray photons passing through the region of interest), and we are able to reconstruct the densities of the region [31]. That is to say, we now have μ. *In turn, this enables the CT scanner to assign density values according to the Hounsfield unit scale of CT numbers for data acquisition/image processing* [35].

Part 3. Summary

The language of mathematics not only permeates all scientific study, but the very application of mathematics itself allows exploration to occur at the limits-of-discovery to find answers to questions that vex human nature.[15] From this perspective we see mathematicians, such as Johann Radon, provide physicists and engineers with "plausible guidance" for their discoveries. This paper presented the first comprehensive essay on the Austrian mathematician Johann Radon (against the backdrop of x-ray CT in medicine). While Radon's contributions to furthering the calculus of variations, measure theory, and functional analysis were significant to mathematics, it is his work on what became known as the Radon transform, widely recognized today thanks in large part to Allan Cormack, which earns Johann Radon an honorary place in the history of

[15] Wininger KL. Applied radiologic science in the treatment of pain: interventional pain medicine. In: Racz GB, Noe CE, eds. Pain Management - Current Issues and Opinions. InTech Publishing. 2012.

medical imaging. Noted here, however, Radon's most significant contributions to the history (of mathematics) may have very well been the part he played in rebuilding the mathematical scene in Austria following the Second World War.

Part 4. References

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[2] Ambrose J. Computerized transverse axial scanning (tomography): Part 2. Clinical application. Br J Radiol 1973;46:1023-1047.

[3] Perry BJ, Bridges C. Computerized transverse axial scanning (tomography): Part 3. Radiation dose considerations. Br J Radiol 1973;46:1048-1051.
- Articles [1], [2], and [3], account for the first clinical write-up on computed tomography.

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- This article references the first professional radiology societies, in England and America.

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- Articles [5], [6], and [7] provide details relative to the first clinical work on computed tomography, conducted by Sir Godfrey Hounsfield and Dr. James Ambrose.

[8] Cormack AM. Representation of a function by its line integrals, with some radiological applications. J Applied Physics 1963;34:2722-2727.

[9] Cormack AM. Representation of a function by its line integrals, with some radiological applications, II. J Applied Physics 1964;35:2908-2913.
- Articles [8] and [9] are the published works of Allan Cormack, describing the theoretical basis of computed tomography.

[10] The Nobel Prize in Physiology or Medicine 1979. Nobelprize.org. Accessed October 1, 2011. Available at http://www.nobelprize.org/nobel_prizes/medicine/laureates/1979/.
- This citation marks the Noble committee's decision to jointly award Sir Godfrey Hounsfield and Allan Cormack the 1979 Nobel Prize in Physiology or Medicine for their work on computed tomography.

[11] Allan M. Cormack - Nobel Lecture. Nobelprize.org. Accessed October 1, 2011. Available at http://www.nobelprize.org/nobel_prizes/medicine/laureates/1979/cormack-lecture.html.
- The narrative of the speech made by Allan Cormack at the 1979 Noble banquet. Here we find his reference to Johann Radon's work on the generalized mathematical foundation for recovery a signal (i.e., function) from its integral (predating Cormacks's own work for his theoretical basis of computed tomography).

[12] Cormack AM. Computed tomography: some history and recent developments. In: Shepp LA, editor. Computed Tomography: Proceedings of Symposia in Applied Mathematics, vol. 27, pp. 35-42, American Mathematic Society; USA. 1983.
- The transcript of the Allan Cormack's talk during a symposium featuring computed tomography, held by the American Mathematical Society. It is within this transcript that we learn the details of Cormack's quest to learn more about the Radon problem.

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- This article recognizes the work of Johann Radon, published just prior to Radon's 100th birthday.

[14] Bockwinkel HBA. On the propagation of light in a biaxial crystal about a midpoint of oscillation. (Dutch) Amst Ak Versl 1906;14:636-651.
- This article marks the first documented description of the Radon problem.

[15] Radon J. Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten (reprint). In: Shepp LA, editor. Computed Tomography: Proceedings of Symposia in Applied Mathematics, vol. 27, pp. 71-86, American Mathematic Society; USA. 1983.
- This article is the published work on the Radon integral by Johann Radon.

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- Article [16] offers a simplistic version of the Radon integral (for historical reference).

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[18] Cramér H, Wold H. Mortality variations in Sweden. Skand Aktuarietidskr 1935;18:161-241.
- Articles [17] and [18] are the works (covering an application of the Radon problem) as discovered by Allan Cormack during his investigation of the Radon problem.

[19] Ossipkov LP. On the jubilee of academician V. A. Ambartsumyan statistical mechanics of stellar systems: From Ambartsumyan onward. Astrophysics 2008;51:428-442.
- Article [19] goes into detail on the work by V. A. Ambartsumyan. The reason this citation is important is because Allan Cormack thought highly of the approach taken to the Radon problem by Ambartsumyan, stating that this approach (a numerical version of the Radon transform) meant that computed tomography could have possibly been developed in an era without computer assistance.

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- This citation defines the calculus of variations.

[21] Eidelman Y, Milman V, Tsolomitis A. Functional Analysis: An Introduction. American Mathematical Society; Providence, RI. 2004. p. xii.

[22] Bronshtein IN, Semendyayev KA, Musiol G, Muehlig H. Handbook of Mathematics. 4th ed. Springer; Heidelberg, Germany. 2004. pp. 594-639.
- Citations [21] and [22] offer insight into functional analysis.

[23] Schmetterer L. Reminiscences to Johann Radon. In: Gindikin S, Michor P, editors. Proceeding and Lecture Notes in Mathematical Physics IV, 75 years of Radon transform. International Press; Cambridge, MA. 1994. pp. 26-28.
- Article [23] gives an account of Johann Radon's personality in his professional career, from the perspective of a former colleague.

[24] Burkovics B. Biography of Johann Radon. In: Gindikin S, Michor P, editors. Proceeding and Lecture Notes in Mathematical Physics IV, 75 years of Radon transform. International Press; Cambridge, MA. 1994. pp. 13-18.
- Article [24] offers a unique encounter with Johann Radon's personal life, from the perspective of his daughter.

[25] Burton DM. Nineteenth Century Contributions: Lobachevsky to Hilbert: The Age of Rigor. In: The History of Mathematics: An Introduction. 7th ed. McGraw-Hill; New York, NY. 2011, p. 614.
- This citation offers a discussion on Karl Weierstrass, the "Father or Modern Analysis."

[26] Funk P. Nachruf auf Prof. Johann Radon. (German) Monatsh Math 1958;62:189-199.
- The obituary of Johann Radon in *Monatshefte für Mathematik*, the journal he reestablished following World War II.

[27] Austrian Academy of Sciences. Johann Radon Institute for Computational and Applied Mathematics. Website homepage. Accessed October 1, 2011. Available at http://www.ricam.oeaw.ac.at/

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- Whereas article [29] is cited with respect to referencing the use of fast Fourier transforms in today's computed tomography scanners, the remaining citations, [30]-[38] reference mathematical concepts relative to the derivation of the Radon transform presented in this paper.

Appendix

A collection of known papers authored by the early 20th century mathematician Johann Radon, compiled from a search of the following databases: 1) Electronic Research Archive for Mathematics maintained by the European Mathematical Society; and 2) MathSciNet maintained by the American Mathematical Society.

1. Radon, J. Gleichgewicht und Stabilität gespannter Netze. (German)
   Arch. Math. (Basel) 5, (1954). 309-316.
   English translation: Equilibrium and stability of strained networks.
2. Radon, J. Zur Polynomentwicklung analytischer Funktionen. (German)
   Math. Nachr. 4, (1951). 156-157.
   English translation: Analytic functions to polynomial.
3. Radon, J. Über geschlossene Extremalen und eine einfache Herleitung der isoperimetrischen Ungleichungen. (German)
   Ann. Mat. Pura Appl. (4) 29, (1949), 315-320.
   English translation: On closed extremal and a simple derivation of isoperimetric inequalities.
4. Radon, J. Zur mechanischen Kubatur. (German)
   Monatsh. Math. 52, (1948). 286-300.
   English translation: For mechanical cubature.
5. Radon, J. Ein einfacher Beweis für die Halbstetigkeit der Integrale der Variationsrechnung auf starken Extremalen. (German)
   Math. Ann. 119, (1944). 205-209.
   English translation: A simple proof of the semicontinuity of integrals of the calculus of variations on strong extremal.
6. Radon, J. Über Tschebyscheff-Netze auf Drehflächen und eine Aufgabe der Variationsrechnung. (German)
   Mitt. Math. Ges. Hamburg 8, (1940). part 2, 147-151.
   English translation: About Chebyshev nets on surfaces of revolution and an object of the calculus of variations.
7. Radon, J. Ein Satz der Matrizenrechnung und seine Bedeutung fr die Analysis. (German)
   Monatsh. Math. Phys. 48, (1939). 198-204.
   English translation: A set of matrix and its importance for the analysis.
8. Radon, J. Bewegungs Invariante Variationsprobleme, betreffend Kurvenscharen. (German)
   Abh. math. Sem. Hansische Univ. 12, (1937). 70-82.
   English translation: Motion invariant variational problems on curves.
9. Radon, J. Singuläre Variationsprobleme. (German)
   Jber. Deutsche Math.-Verein. 47, (1937). 220-232.
   English translation: Singular variational problems.
10. Radon, J. Annäherung konvexer Körper durch analytisch begrenzte. (German)
    Monatsh. Math. Phys. 43 (1936), no. 1, 340-344.
    English translation: Analytical approximation of convex bodies by limit.
11. Radon, J. Restausdrücke bei Interpolations- und Quadraturformeln durch bestimmte Integrale. (German)
    Monatsh. Math. Phys. 42 (1935), no. 1, 389-396.
    English translation: Residual terms for interpolation and quadrature formulas by certain integrals.
12. Radon, J. Bestimmung einer Riemannschen Metrik durch Krümmungseigenschaften. (German)
    Monatsh. Math. Phys. 35 (1928), no. 1, 9-24.
    English translation: Determination of a Riemannian metric by curvature properties.

13. Radon, J. Mathematik und Wirklichkeit. (German)
Sitzungsberichte Erlangen 58/59, (1928). 181-190.
English translation: Mathematics and Reality.

14. Radon, J. Zum Problem von Lagrange. 4 Vorträge, gehalten im Mathematischen Seminar der Hamburgischen Universitt (7.-24. Juli 1928). (German)
Abhandlungen Hamburg 6, (1928). 273-299.
English translation: On the problem of Lagrange.

15. Radon, J. Über die Oszillations theoreme der konjugierten Punkte beim Probleme von Lagrange. (German)
Sitzungsberichte München 1927, (1927). 243-257.
English translation: On the oscillation of the conjugate points theorems in problems of Lagrange.

16. Radon, J. Über konforme Geometrie. VI: Kurvennetze auf Flchen und im Raume von Riemann. (German)
Abhandlungen Hamburg 5, (1926). 45-53.
English translation: On conformal geometry. VI: curve networks on land and in space of Riemann.

17. Radon, J. Über konforme Geometrie. V: Neue Kennzeichnung der zyklischen Kurvennetze. (German)
Abhandlungen Hamburg 4, (1926). 313-320.
English translation: On conformal geometry. V: New labeling the cyclic curve networks.

18. Radon, J. Berichtigung zu der Abhandlung "Zur Behandlung geschlossener Extremalen in der Variationsrechnung". (German)
Abhandlungen Hamburg 4, (1925). 13-14.
English translation: Correction to the paper "For the treatment of closed extremals in the calculus of variations".

19. Radon, J. Zur Riemannschen Geometrie. (German)
Jahresbericht D. M. V. 33, (1925). 95-96 kursiv.
English translation: To the Riemann geometry.

20. Radon, J. Zur Behandlung geschlossener Extremalen in der Variationsrechnung. (German)
Hamb. Abh. 1, (1922). 195-205.
English translation: For the treatment of closed extremals in the calculus of variations.

21. Radon, J. Lineare Scharen orthogonaler Matrizen. (German)
Hamb. Abh. 1, (1921). 1-14.
English translation: Linear bands of orthogonal matrices.

22. Radon, J. Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten. (German)
Math. Ann. 83 (1921). no. 1-2, 113-115.
English translation: Volumes of convex bodies that contain a common point.

23. Radon, J. Über die Bestimmung einer Riemannschen Metrik aus dem Krümmungstensor. (German)
Deutsche Math.-Ver. 30, (1921). 76.
English translation: Identification of a Riemannian metric from the curvature tensor.

24. Radon, J. Über statische Gravitationsfelder. (German)
Hamb. Abh. l, (1922). 268-280.
English translation: About static gravitational fields.

25. Radon, J. Über die Randwertaufgaben beim logarithmischen Potential. (German)
Wien. Anz. 56, 190; Wien. Ber. (2) 128, (1920). 1123-1167.
English translation: On the boundary value problems in logarithmic potential.

26. Radon, J. Über lineare Funktionaltransformationen und Funktionalgleichungen. (German)
Wien. Anz. 56,189; Wien. Ber. (2) 128, (1919). 1083-1121.
English translation: About linear functional transformations and functional equations.

27. Radon, J. Über affine Geometrie XVII: Zur Affine geometrie der Regelflächen. (German)
Leipz. Ber. 70, (1919). 147-155.
English translation: On affine geometry XVII: The affine geometry of ruled surfaces.

28. Radon, J. ber affine Geometrie XVI: Die Grundgleichungen der affinen Flchentheorie. (German)
Leipz. Ber. 70, (1918). 91-107.
English translation: On affine geometry XVI: The basic equations of the affine surface theory.

29. Radon, J. Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten. (German)
Leipz. Ber. 69, (1917). 262-277.
English translation: On the definition of functions by their integral values along certain manifolds.

30. Radon, J. Über eine Erweiterung des Begriffes der konvexen Funktionen mit einer Anwendung auf die Theorie der konvexen Körper. (German)
Wien. Ber. 125, (1916). 241-258.
English translation: An extension of the concept of convex functions with an application to the theory of convex bodies.

31. Radon, J. Die Kettenlinie bei allgemeinster Massenverteilung. (German)
Wien. Ber. 125, (1916). 221-240.
English translation: The chain line in the most general mass distribution.

32. Radon, J. Über eine besondere Art ebener konvexer Kurven. (German)
Leipz. Ber. 68, (1916). 123-128.
English translation: Through a special kind of plane convex curves.

33. Radon, J. Theorie und Anwendungen der absolut additiven Mengenfunktionen. (German)
Wien. Ber. 122, (1913). 1295-1438.
English translation: Theory and applications of absolutely additive set functions.

34. Radon, J. Zur Theorie der *Meyer* schen Felder beim *Lagrange* schen Variationsproblem. (German)
Wien. Ber. 120, (1911). 1337-1360.
English translation: On the theory of *Meyer* electromagnetic fields for *Lagrangian* between variational problem.

35. Radon, J. Über einige Fragen betreffend die Theorie der Maxima und Minima mehrfacher Integrale. (German)
Monatsh. Math. Phys. 22 (1911). no. 1, 53-63.
English translation: On some questions concerning the theory of maxima and minima of multiple integrals.

36. Radon, J. Über das Minimum des Integrals $\int_{s_0}^{s_1} f(x, y, \theta, \kappa)ds$. (German)
Wien. Ber. 119, (1910). 1257-1326.
English translation: On the minimum of the integral $\int_{s_0}^{s_1} f(x, y, \theta, \kappa)ds$.

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Harmonic Analysis on Euclidean Spaces		
Harmonic Analysis in One Variable	Harmonic Analysis in Several Variables	Nontrigonometric Harmonic Analysis
Trigonometric polynomials, inequalities, extremal problems	Fourier series and coefficients	Orthogonal functions and polynomials, general theory
Trigonometric approximation	Summability	Fourier series in special orthogonal functions
Trigonometric interpolation	Fourier and Fourier-Stieltjes transforms and other transforms of Fourier type	General harmonic expansions, frames
Fourier coefficients, Fourier series of functions with special properties	Multipliers	Other transformations of harmonic type
Convergence and absolute convergence of Fourier and trigonometric series	Singular and oscillatory integrals	Uniqueness and localization for orthogonal series
Summability and absolute summability of Fourier and trigonometric series	Maximal functions, Littlewood-Paley theory	Completeness of sets of functions
Trigonometric series of special types	H^p-spaces	Wavelets and other special systems
Fourier and Fourier-Stieltjes transforms and other transforms of Fourier type	Function spaces arising in harmonic analysis	
Multipliers	Harmonic analysis and Partial Differential Equations	
Conjugate functions, conjugate series, singular integrals		
Lacunary series of trigonometric and other functions; Riesz products		
Probabilistic methods		
Uniqueness of trigonometric expansions, uniqueness of Fourier expansions, Riemann theory, localization		
Completeness of sets of functions		
Trigonometric moment problems		
Classical almost periodic functions, mean periodic functions		
Positive definite functions		
Convolution, factorization		

Table 1. Mathematics Subject Classification on Harmonic Analysis—per the American Mathematical Society. The Radon problem, which relates to reconstructing a function from its integral, falls under the second arm, "Harmonic Analysis in Several Variables," and within this subclassification, belongs under "Fourier and Fourier-Steiltjes transforms and other transforms of Fourier type."

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Radon transform	Denoted here by the script letter ℛ, **the Radon transform is a special application of the Fourier transform.**
Fourier transform	The Fourier transform (script letter ℱ) is known as a limiting case of the Fourier series, concerned with analysis (in the general sense of the word "analysis") of nonperiodic phenomena. ***Instrumentally, the study of the Fourier transform is a mathematical cornerstone for signal processing, and therefore, contains both a spatial domain (we will call "r") and a frequency domain, φ (phi).***
Inverse relation: analysis and synthesis	Whereas analysis deals with breaking up signals (i.e., functions) into constituent parts, synthesis is concerned with re-assembling or reconstructing a signal from its constituent parts. Both analysis and synthesis are accomplished by the linear operations, series and integrals, respectively.
∫ (***integral symbol***)	The symbol for integration, one of two main operations in calculus, which is used to accumulate [the values of] variables, the **integrand**.
Sine function	For our purposes, denoted as (sin φ). Sine is a simple repetitive mathematical (trigonometric) function, and can be used to model periodic phenomena, such as signals or functions. In the unit circle (a circle with a radius of one [1]), the sine value of a variable, such as angle φ, equates to the y-coordinate, expressed as sin(φ) = y. (Note: in the model, the x_2-variable and the x_2-axis will be the names given to the y-variable and y-axis, respectively.)
Cosine Function	For our purposes, denoted as (cos φ). Sine is a simple repetitive mathematical (trigonometric) function, and can be used to model periodic phenomena, such as signals/functions. In the unit circle (a circle with a radius of one [1]), the cosine value of a variable, such as angle φ, equates to the x-coordinate, expressed as cos(φ) = x. (Note: in the model, the x_1-variable and the x_1-axis will be the names given to the x-variable and x-axis, respectively.)
Vectors and the (unit normal vector)	Vectors are quantities which have magnitude (length) and direction; graphically depicted as arrows and denoted in **bold** type face. The unit normal vector, written as **n** = ‹**cos(φ),sin (φ)**›, has length one (1), perpendicular to a line.
μ (mu)	Denotes the signal/function that represents the densities in the imaged region we are trying to recover.
ρ (rho)	Represents a signed distance (i.e., either positive or negative) in a family of lines (i.e., x-ray trajectories in the exit beam), relative to the line through the origin and the direction of the unit normal vector.
φ (phi)	Denotes the angle of the unit normal vector (and thus, any line in the family of lines) to the x_1-axis.
δ (delta)	Represents the Dirac delta function for a line impulse, and is important when taking the line integral (i.e., line or trajectory of x-rays in the exit beam).
ξ (Xi)	Denotes a non-finite (or infinite) measurement, such as infinite distance.
∞ (infinity symbol)	Denotes a non-finite (or infinite) measurement, such as infinite distance.
2π	Represents the number "2π", the circumference of a circle. (We will use "2π" in our version of Euler's formula, see below, for the model.)
"i"	The letter "i" is used to represent the imaginary numbers, whereas when such numbers are mixed with "real" numbers, the resultant term is named a complex-number or function or exponential. For our purposes, "i" equals the square root of minus one (expressed as **i** = √(-1)). (For more on "i", see Euler's formula.)
Euler's formula **$e^{i\varphi} = \cos(\varphi) + i \sin(\varphi)$**	Euler's formula (named after Leohnard Euler) refers to the complex exponential, "'e' to the iφ equals cos(φ) plus "i" times sin(φ)", where "e" is the base of the natural logarithm, "i" is the imaginary number, and cosine and sine are the trigonometric functions. *In the model, we will use a version of Euler's formula, written as follows: $e^{-2\pi i r \rho}$ or "e^-2πirp". This version will be an elementary constituent for our application of the Fourier transform.*

Table 2. Glossary of terms.

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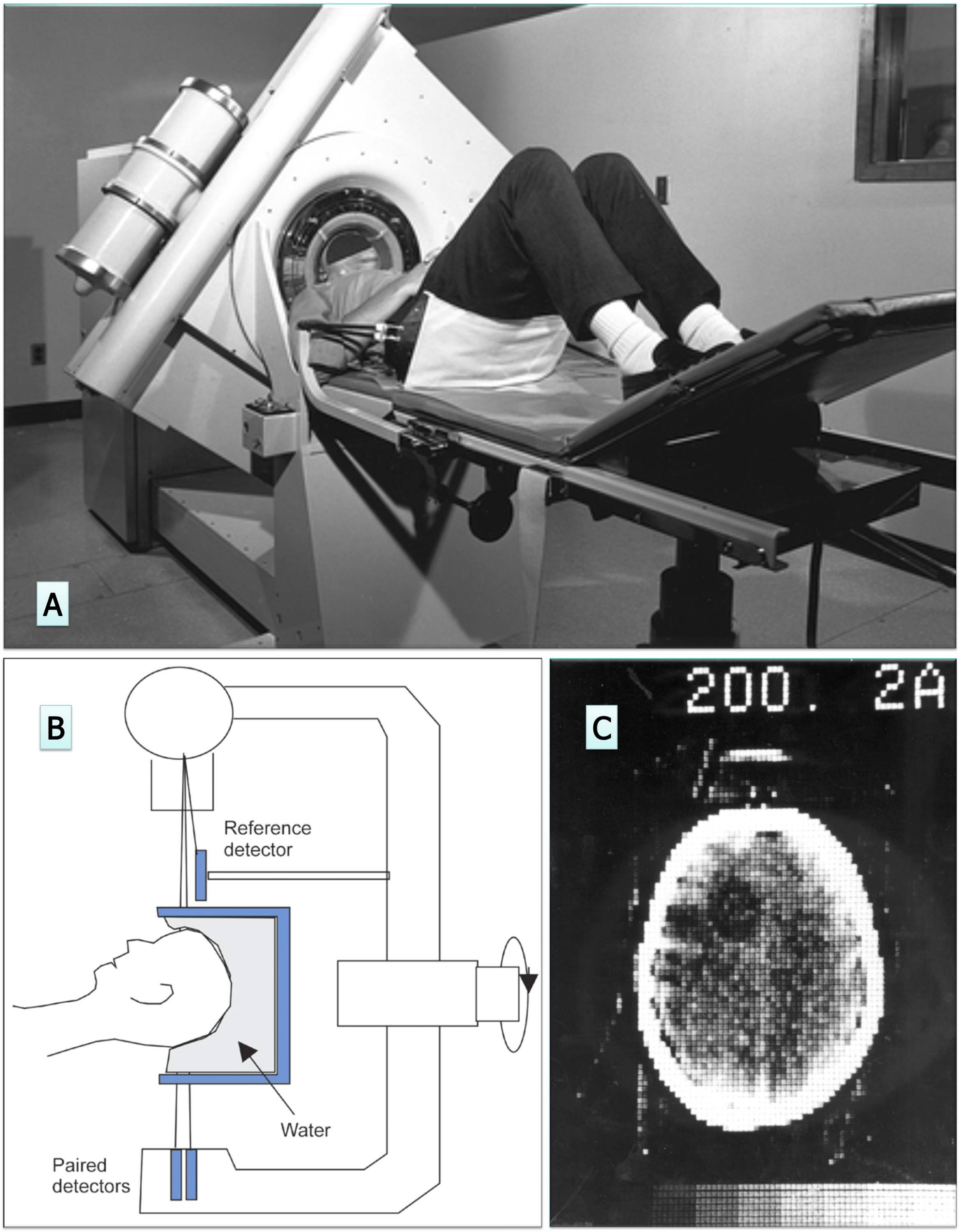

FIGURE 1. (A) The prototype CT scanner developed by Sir Godfrey Hounsfield and built by Electric & Musical Industries, Ltd. (EMI) 1971. (B) Notably, the scanner was a dedicated head scanner (a whole body scanner was built by Hounsfield and EMI in 1975). (C) Confirmed frontal lobe tumor seen in the first patient scanned on the prototype EMI scanner at Atkinson Morley Hospital on 1st October 1971.

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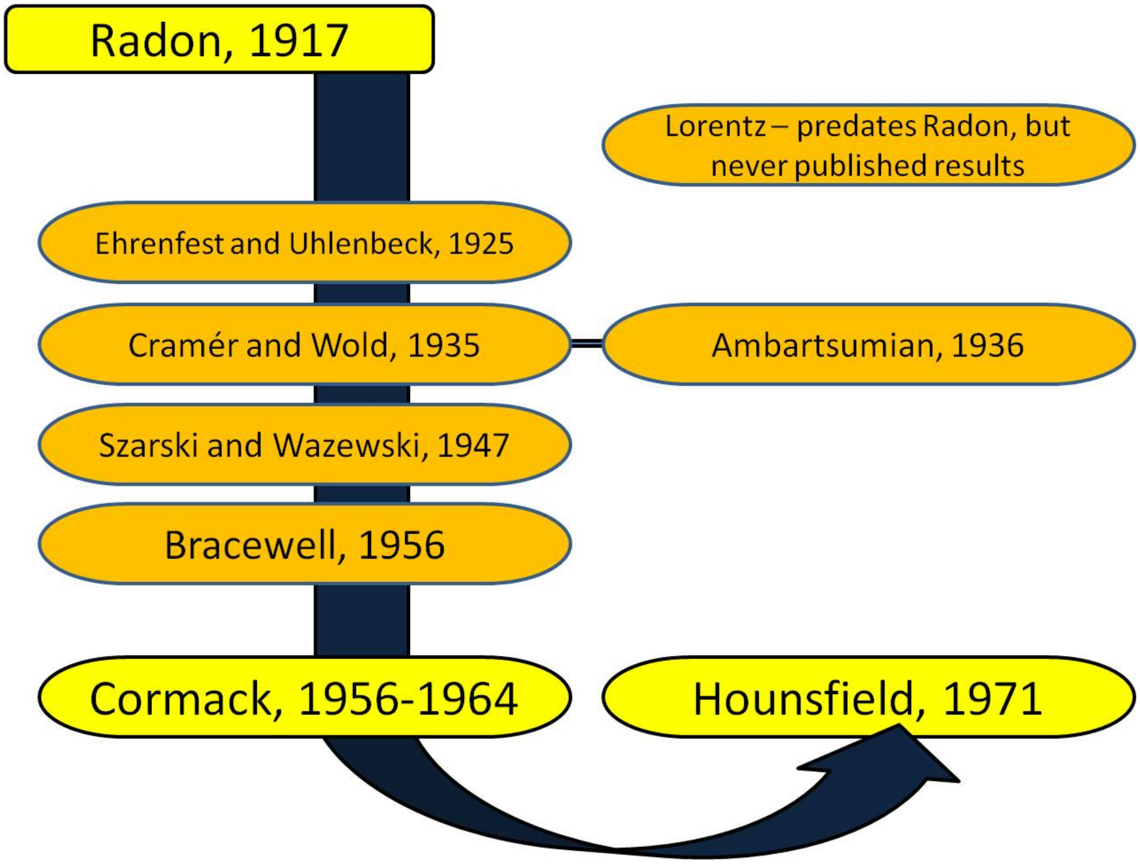

Figure 2. Graphical representation of the lineage of the Radon transform.

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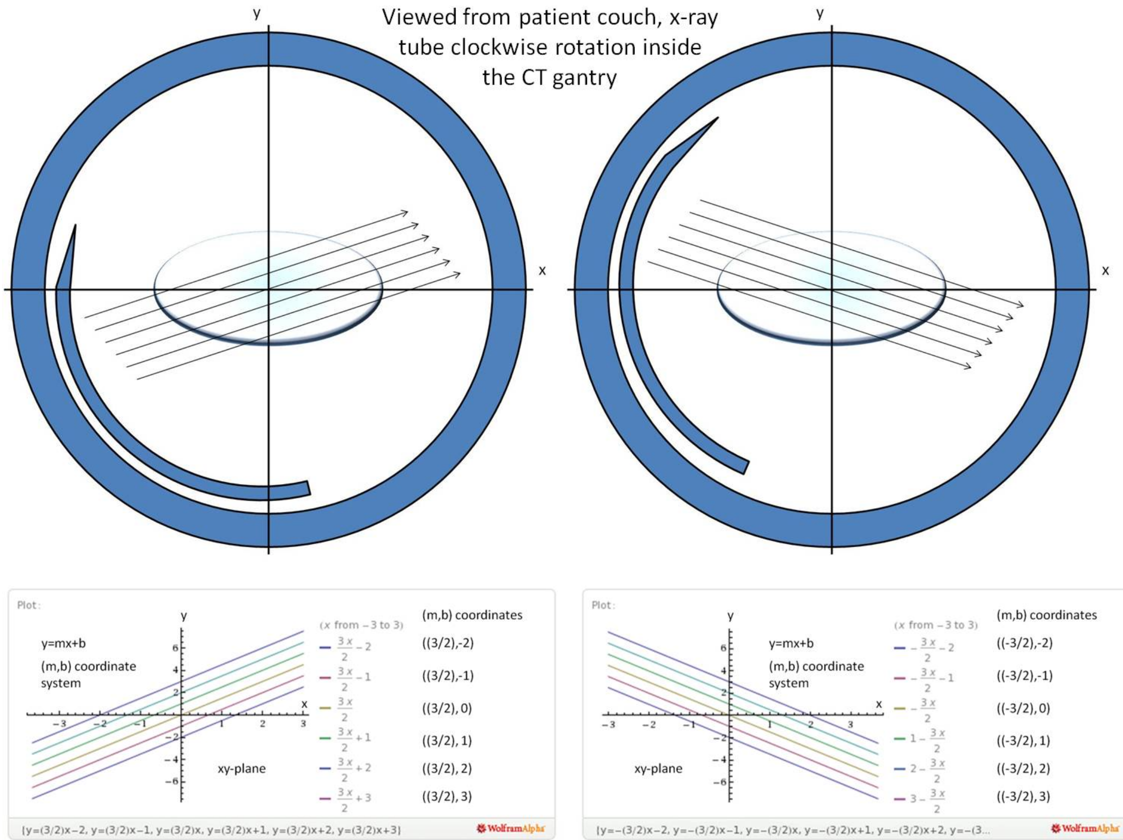

FIGURE 3. (Top left) Gantry of CT scanner showing trajectories of x-rays [long arrows] emitted as lines/family of parallel lines. (Top right) Illustrated clockwise rotation of the x-ray tube inside the gantry. (Bottom left/right) Representative, corresponding line equations written in the slope-intercept form $y = mx + b$ in the xy-plane. The (m, b) coordinates are given by the line equations.

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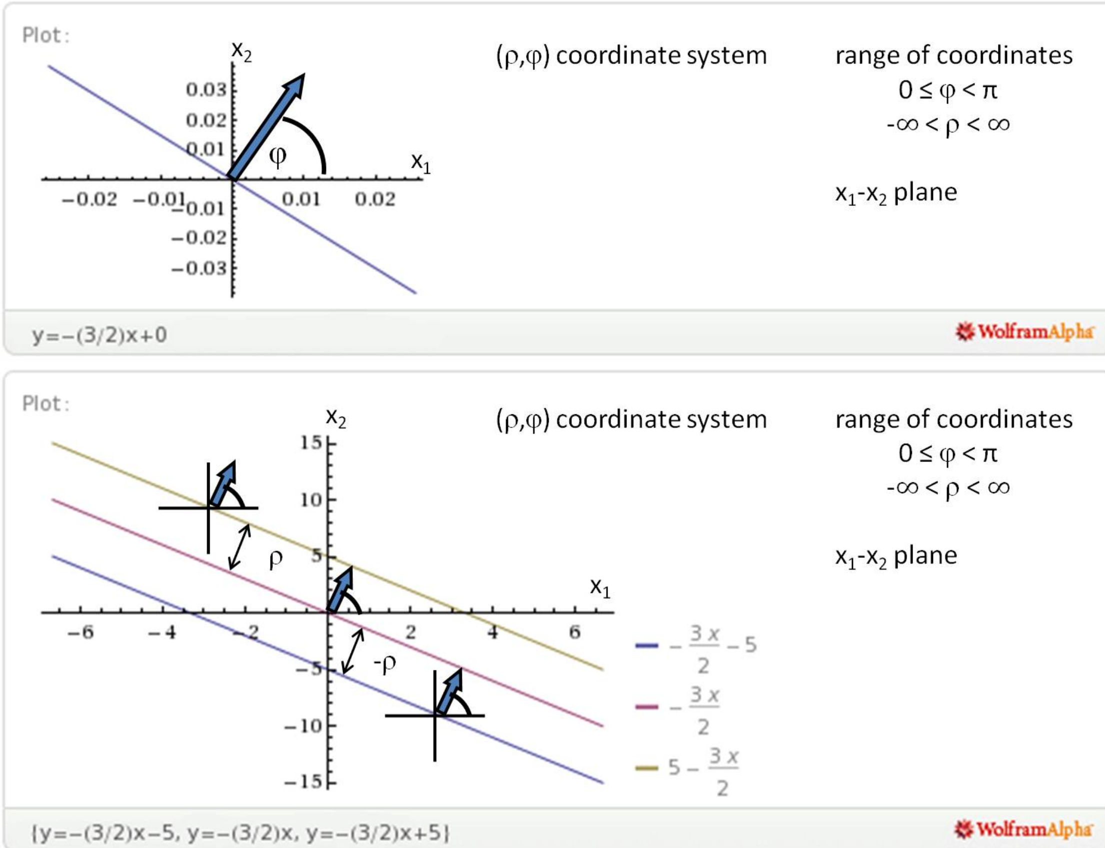

Figure 4. A better-suited coordinate system (ρ, φ) for the model. (Top panel) Arrow demonstrating the unit normal vector associated with the line passing through the origin, and oriented with an angle phi (φ) to the x_1-axis in the x_1, x_2-plane. (Bottom panel) A family of 3-parallel lines and their unit normal vectors [arbitrarily placed on the lines] showing signed distances rho (ρ) from the origin [double ended arrows]. By convention, distances are positive [i.e., positive rho (ρ)] when measured in the direction of the normal vector from the line passing through the origin to associated parallel lines. In a similar fashion, distances are negative [i.e., negative rho $(-\rho)$] when measured from the line passing through the origin to parallel lines spatially existing opposite to the direction established for the normal vector. Rho (ρ) is zero at the line passing through the origin. (Note: the unit normal vectors are not drawn to scale, and when compared to Figure 3, the xy-plane has been renamed the x_1, x_2-plane.)

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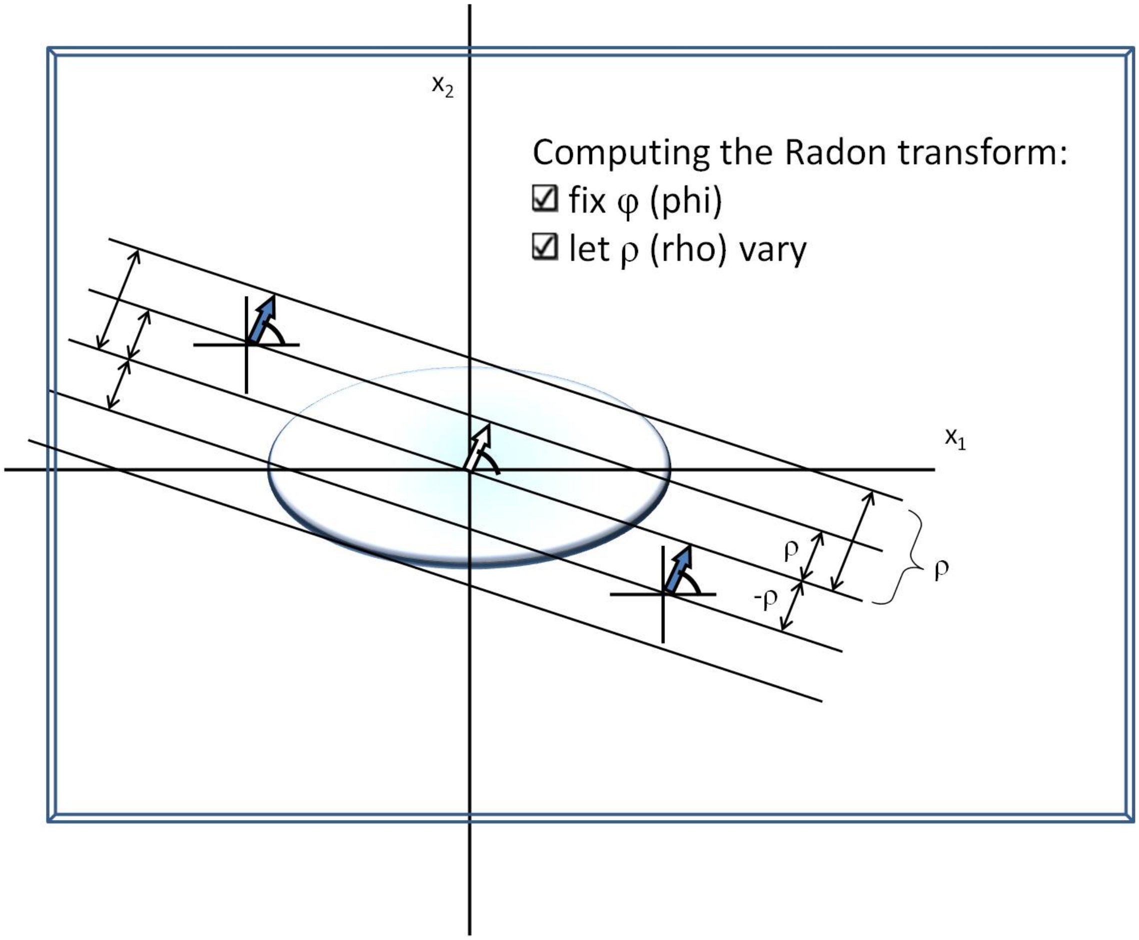

FIGURE 5. Notice what this family of parallel lines has in common, each line has the same angle φ (phi) to the x_1-axis. Thus, the variable to use with respect to integrating the constituent integrals of the Radon transform (and those integrals contributing to the Fourier transform) is the distance ρ (rho) of a line from the line passing through the origin.