Rick Hill

Thomas Harriot's *Artis Analyticae Praxis* and the Roots of Modern Algebra

University of Missouri – Kansas City
Supervising Instructor: Richard Delaware

r_hugel@hotmail.com

5419 Walmer St., Mission KS 66202

Thomas Harriot's *Artis Analyticae Praxis* and the Roots of Modern Algebra

Thomas Harriot (c. 1560 – 1621) was an Englishman at the forefront of the political, scientific, and mathematical developments of his time. He participated in an expedition to North America to chart the newly formed colonies (1585 – 86), was a highly accomplished navigator, surveyor, and translator, discovered the sine law of refraction almost 20 years before Snellius (1602 compared to 1621), was as talented and accomplished an astronomer as Galileo (1564 – 1642) and Kepler (1571 – 1630), and was imprisoned along with his patron, Henry Percy, the Ninth Earl of Northumberland (1564 – 1632) for suspicion of being involved with the Gunpowder Plot (1605), a failed assassination attempt against King James I.

His contributions to mathematics lie chiefly in his advancement of algebra. Only one work of Harriot's mathematics was ever published, the *Artis Analyticae Praxis (The Practice of the Analytical Art),* which appeared posthumously, in 1631, ten years after his death. It was edited and assembled by one of his friends and executors, William Warner.

The *Praxis* is "a work on the theory of equations and solution of numerical polynomial equations." [2, p. 153], and it contained three primary achievements:

1. The creation of a clear notation, which led to
2. Manipulating equations while working in purely symbolic forms, which led to
3. Understanding of the structure of polynomials.

The first two helped provide the means for mathematicians to work with certainty and efficiency. But the last became "the foundation of all subsequent work on polynomial equations, and [was] to help lead eventually to the development of modern abstract algebra." [7, p. 124; all comments in square brackets are mine.]

Harriot's inspiration and model for the *Praxis* was Francois Viete's *In artem analyticen isagoge (Introduction to the art of analysis)*. Viete's "contribution to algebra lies in the new level of generality engendered by his notation . . . [however], the link with a geometrical base is never broken." [4, p.8] While the "concepts [in the *Praxis*] are rooted in *Isagoge* . . . [Harriot] transcend[s] the usage of Viete and give[s] these terms an exclusively **algebraic** meaning." [4, p. 211].

Harriot first lays out examples of the basic four operations of symbolic arithmetic – addition, subtraction, multiplication, and division. In his notation, as in Viete's, **vowels (such as *a*) signify unknowns and consonants (*b, c*) signify known positive numbers.** The examples illustrate the clarity of his notation and the process of working with purely symbolic forms. One obvious difference between Harriot's notation and modern notation is the absence of exponents. Rather than writing a^2 or a^3, he writes *aa* and *aaa*.

Here are examples of the four operations:

<u>Addition:</u>

To be added	*a*	*aa*	*aaa*
	b	*bc*	*bcc*
Total	*a + b*	*aa + bc*	*aaa + bcc*

<u>Subtraction:</u>

Given	a	$a + b$	$aa + cc$
To be subtracted	b	$c + b$	$aa + cc$
Remainder	$a - b$	$a - c$	0

<u>Multiplication:</u>

To be multiplied	$b - a$ $b - a$	$b + a$ $b - a$
	$bb - ba$ $- ba + aa$	$bb + ba$ $- ba - aa$
Product	$bb - 2ba + aa$	$bb \quad - \quad aa$

The algorithm for multiplication reads this way: each term of the first row, in this case $b - a$ or $b + a$, is multiplied by the first term of the second row, in this case b. The next line consists of each term of the first row multiplied by the second term of the second row, - a. These rows are stacked so that like powers of the unknown, in this case *a,* align, which permits easy addition by combining like terms.

<u>Division:</u>

Dividend	$bbcc$	$ba + ca + da$
Divisor	cc	a
Quotient	bb	$b + c + d$

Here are additional examples of division, in which Harriot shows how to deal with rational numbers and which seem entirely modern:

$$\frac{bca}{b} = ca$$

and

$$\frac{ac}{b} + \frac{dd}{g} = \frac{acg}{bg} + \frac{bdd}{bg} = \frac{acg + bdd}{bg}.$$

After showing how the symbolic arithmetic works, Harriot showed what kinds of equations would be created if binomial factors were multiplied by each other. He called these kinds of equations “originals” (3, p. 94).

Here is one example of how to generate such an “original” equation, which is the product of two binomial factors:

$$\left.\begin{array}{l} a - b \\ a - c \end{array}\right| \begin{array}{l} \\ = aa - ba \\ \quad - ca + bc \end{array}$$

Here is another example of an “original equation,” which is the product of three binomial factors:

$$\left.\begin{array}{l} a + b \\ a + c \\ a - d \end{array}\right| \begin{array}{l} \\ \\ = aaa + baa + bca \\ \quad + caa - bda \\ \quad - daa - cda - bcd \end{array}$$

The following diagram illustrates the multiplication algorithm for generating the product of three binomial factors:

$a + b$ $a + c$ $a - d$	aaa	$a + b$ $a + c$ $a - d$	baa	$a + b$ $a + c$ $a - d$	bca		
		$a + b$ $a + c$ $a - d$	caa	$a + b$ $a + c$ $a - d$	$-bda$		
		$a + b$ $a + c$ $a - d$	$-daa$	$a + b$ $a + c$ $a - d$	$-cda$	$a + b$ $a + c$ $a - d$	$-bcd$

The primary task of the algorithm is to ensure every possible product of three terms is generated. Imagine the binomial factors are ledges and the lines which connect the terms to be multiplied are ramps; then each diagram above shows the path a marble could take as it fell from one ledge to the next. As a product is generated, a potential path has been used. And one continues until all possible paths are exhausted.

The secondary task of the algorithm is to simplify the process of combining like terms once we have generated all the products. This is accomplished by recognizing that the like terms to be combined will have common powers of the unknown variable, *a*, as was the case with the quadratic product. Since we want to combine these terms using addition, we generate our products so as to create rows where no row has terms with common powers of *a,* and then we stack our rows with common powers of *a* from each row aligned in columns.

Making use of the inherent order of the terms we've chosen for our known constants, *b,c,d,* helps us keep track of paths as we generate them. So, for instance, the first product, *aaa,* shows our marble falling straight down. This is the only path that is a product of all three *a*'s. But we will have three potential paths that use two *a*'s, each with either *b,c,* or *d*. We go to *b* first – it naturally precedes *c* and *d* -- then fall to *a,* then *a* again; hence, *baa*. Since we've used the only path with three *a's* and one with two *a*'s, we will now complete this row by seeking a path that uses only one *a.* Again, we start at *b,* but now we fall to *c,* and then to *a;* hence, *bca*. And our first row is complete.

And so it goes, row by row, until all possible paths have been determined.

After showing how to generate "original" equations, Harriot focused on deriving "canonical" equations "from their originals." These equations were called "canonical" because "they are adapted to canons or rules for finding numeral roots." (3, p. 94) Put another way:

> "A Canonical Equation was one expressed in a standard form produced by the multiplication of binomial factors. What Harriot did may be summarised (for a quadratic equation) in modern notation, as follows:
>
> If a is a root of the equation, $x = a$
>
> Then, $x - a = 0$
>
> If b is a root, $x = b$
>
> Then, $x - b = 0$
>
> It follows that $(x - a)(x - b) = 0$
>
> i.e., $x^2 - (a + b)x + ab = 0$
>
> and $x^2 - (a + b)x = -ab$
>
> The expression in the final line is the **Canonical Equation** and it follows that any equation in such a form has roots *a, b*. (4, p. 5) . . . Harriot has been credited with being

the first to equate all the terms of an equation to zero. He may have been able to do this because there is less likelihood of seeing the side ‘opposite’ to the zero as ‘something’ equated to ‘nothing’ when the entire equation is seen abstractly and symbolically. However, Harriot never leaves the equation in this form and a final line is always added equating all to the “homogene” [the constant term] even if this is negative.” (4, p. 13)

Here is an example from early in the *Praxis* of how Harriot derived a canonical equation that is a quadratic (remember, ***a* is the variable** and ***b,c* are the constants**):

Section 2, Proposition 2

The canonical equation $\begin{array}{l} aa-ba \\ \quad -ca = -bc \end{array}$ is derived from the original equation

$$\left.\begin{array}{l} a-b \\ \underline{a-c} \end{array}\right| = \begin{array}{l} aa-ba \\ -ca+bc \end{array}$$

by putting *b* or *c* equal to *a.*

[Proof]

For if we put $a = b$, it will be true that $a - b = 0$; or if $a = c$, then it will be true that $a - c = 0$. Consequently, putting $a = b$ or *c*, it will be true that

$$\left.\begin{array}{l} a-b \\ \underline{a-c} \end{array}\right| = 0.$$

Moreover, it is true by multiplying out that

$$\left.\begin{array}{l} a-b \\ \underline{a-c} \end{array}\right| = \begin{array}{l} aa-ba \\ -ca+bc \end{array}$$

which is the original equation designated in the proposition.

Therefore, $\begin{array}{l} aa-ba \\ \quad -ca+bc = 0. \end{array}$

Therefore, $aa - ba$

$- ca = -bc$ which is the proposed equation.

Thus the proposed canonical equation is derived from the designated original equation by putting *b* or *c* equal to *a,* as was stated in the proposition. [4, p. 17]

∎

By setting *b* or *c* equal to *a*, then either factor, *a – b* or *a – c*, was equal to zero. This would make the resulting polynomial equal to zero. With the polynomial equal to zero, it was possible to use the arithmetic operations to arrange an equation whereby all the unknown terms were on one side of the equal sign and the sole known constant term was on the other. The resulting equation,

$$aa - ba - ca = -bc$$

stood as a model of its type, the quadratic canonical.

After creating the many different permutations of canonical equations, Harriot demonstrated how the value of the unknowns could be found, based on the form of the canonical. Here is an example for finding the unknowns of a quadratic canonical equation:

Section 4, Proposition 2

b or *c* are roots of the equation $aa - ba$

$- ca = -bc,$

equal to the sought root *a*.

[Proof]

For if *b* is put equal to the root *a* of the equation

$$aa - ba$$
$$- ca = -bc,$$

then changing *a* into *b* we get

$$bb - bb$$
$$- cb = -bc$$

But the equality here is obvious.

Therefore, having put *b* equal to the root *a,* it is indeed (shown to be) equal. Likewise, if *c* is put equal to the root *a*, then changing *a* into *c*, we get

$$cc - bc$$
$$- cc = -bc$$

And the equality is also obvious here.

Therefore having put *c* equal to the root *a*, it is indeed (shown to be) equal.

Thus *b* and *c* equal to the sought root *a,* as was stated in the proposition.

$\blacksquare$

And in the following

Lemma

it is demonstrated that

[Statement:] there is no other root besides *b* and *c* equal to the root *a* of the equation.

[Proof of Lemma]

Suppose another root of the equation equal to the root *a* and not equal to the roots *b* and *c* could be given; let it be *d* (or anything else).

Then putting $d = a$ it will be true that $dd - bd$

$$- cd = -bc.$$

Therefore [putting *bd* on the right] $dd - cd = +bd - bc$

Therefore $\begin{array}{r|cr|} +d-c & & +d-c \\ \underline{\quad d} & = & \underline{\quad b} \end{array}$ [meaning $(d-c)d = (d-c)b$]

Therefore $d = b$ which is contrary to the hypothesis.

Or it will be true that [putting instead *cd* on the right] $+dd - bd = +cd - bc$.

Therefore $\begin{array}{r|cr|} +d-b & & +d-b \\ \underline{\quad d} & = & \underline{\quad c} \end{array}$ [meaning $(d-b)d = (d-b)c$]

Therefore $d = c$ which is contrary to the hypothesis.

Therefore it is not true that $d = a$ as was assumed. This can be demonstrated in a similar way for any other root apart from *b* and *c.* [4, p. 53]

$\blacksquare$

Here is another example, which is for finding the unknowns of a cubic canonical equation:

Section 4, Proposition 3

d is the root of the equation $\begin{aligned} &aaa + baa + bca \\ &+ caa - bda \\ &- daa - cda = +bcd \end{aligned}$ equal to the sought root *a*.

[Proof:]

For if *d* is put equal to the root *a* of the equation $\begin{aligned} &aaa + baa + bca \\ &+ caa - bda \\ &- daa - cda = +bcd \end{aligned}$, then changing *a* into *d* we get $ddd + bdd + bcd$

$$\begin{aligned} &ddd + bdd + bcd \\ &+ cdd - bdd \\ &- ddd - cdd = +bcd\ . \end{aligned}$$

But, once terms of opposite signs have been removed, the equality here is obvious. Therefore having put *d* equal to root *a,* it is indeed (shown to be) equal.

And in the following $\blacksquare$

Lemma

it is demonstrated that

[Statement:] there is no other root besides *d* equal to the root *a* in the equation.

[Proof of Lemma]

Suppose another root of the equation equal to the root *a* and not equal to the root *d* could be given; let it be *b* or *c* (or anything else).

Then putting *c* = *a* it will be true that

$$\begin{aligned} &ccc + bcc + bcc \\ &+ ccc - bdc \\ &- dcc - cdc = +bcd \end{aligned}$$

And on rearranging the terms $+\,2ccc + 2bcc = +2ccd + 2\ bcd$.

Therefore $\begin{array}{c|} +cc+bc \\ \hline \end{array}\quad c \quad = \begin{array}{c|} +cc+bc \\ \hline d \end{array}$ [meaning $(cc+bc)c = (cc+bc)d$]

Therefore, $c = d$ which is contrary to the hypothesis.

Therefore, it is not true that $c = a$ as was assumed. And by precisely the same reasoning the same thing can be concluded about *b* or any other root apart from *d.* [4, 54]

$\blacksquare$

The examples shared so far come from the first part of the *Praxis,* which concerns itself with the theory of equations. In the first part, canonical equations are generated of the quadratic, cubic, quartic, and even quintic orders. The second part of the *Praxis* gives examples of solving polynomial equations with numerical coefficients by using a method of successive approximation. In its overall organization, the *Praxis* resembles (or maybe, better said, prefigures) the organization of a modern textbook, with theory followed by examples.

Establishing Harriot's place in the history of mathematics is difficult for several reasons. The first and most important was that he didn't publish during his lifetime. The *Praxis* is as much a creation of William Warner's as it is Harriot's. Warner took it upon himself to finally put it together after another friend and executor who was charged with the task, William Torporley, had dawdled for 10 years following Harriot's death [6, pp. 4 – 7]. While Warner's enthusiasm for his friend's work and commitment to preserve his legacy were admirable (he published the *Praxis* at his own expense), he did not share Harriot's mathematical talents, and the *Praxis* is a lesser work as a result [4, pp. 11 – 13; 2, p. 164].

The most glaring example of how the *Praxis* was diminished is the handling of negative roots:

> "The *Praxis,* apart from two examples . . . recognizes only positive roots . . . [T]he issue with regard to negative roots is not existence but utility. (Yet negative numbers . . . are not rejected. Many equations are written with the left-hand side equated to a negative quantity on the right.) The manuscripts, on the other hand, present a very different picture, with virtually universal recognition of negative roots." [2, p. 167]

Harriot's papers also show he was aware that complex and imaginary numbers could be roots, but there is no record of how he regarded such numbers [2, p. 170].

Another difficulty stems from the peripatetic history of Harriot's papers after his death. Once Harriot's benefactors, friends, and executors had passed away, the direct connection to his papers was lost. The papers were scattered among the heirs of benefactors to whom they were bequeathed and then raided by well-intentioned but ultimately unproductive mathematicians seeking to publish Harriot's work. The collection was permanently divided into several pieces, and Harriot's organization of the material was lost. It wasn't until the 1960s that a concerted effort was begun to thoroughly explore what mathematics lay in Harriot's papers and correlate the *Praxis* with what survived of them [2, p. 155].

As Harriot did not speak for himself, through publication, and his papers were unable to speak for him, because of being scattered or lost, his reputation rested on the *Praxis* and those who had access to (some of) his work. John Wallis, in particular, was a stout defender of Harriot's reputation, and came to believe that Descartes (1596 – 1650) unjustly laid claim to work that Harriot had already done. Unfortunately, Wallis's assertions were undermined by a lack of documentary evidence and the taint of thinly-veiled nationalist pride [5, pp. 117 - 125].

Despite his slim published legacy, Harriot "had the highest reputation among his contemporaries" and his influence "on later English mathematicians such as John Pell (1611 – 1685), Charles Cavendish (1592 – 1664), and . . . John Wallis (1616 – 1703) was considerable." (4, p. 4).

Harriot's contribution and influence are perhaps best summarized in this way:

"There is a reciprocal relation between symbolism and mathematical thought-processes, and it would be hard to overestimate the effect of Harriot's techniques and clarity of thought expressed in a symbolism that directs what you do visually and therefore makes mathematics accessible in a totally new way . . . The visualizability is obvious but profoundly important." [7, p. 184]

References:

1. **The Number System of Algebra Treated Theoretically and Historically,** Henry Burchard Fine, Leach, Shewell and Sanborn, 1890
2. **Thomas Harriot, An Elizabethan Man of Science,** ed. by Robert Fox, Ashgate Publishing Ltd., 2000
3. **A Philosophy and Mathematical Dictionary,** Charles Hutton, self-published, 1815
4. **Thomas Harriot's Artis Analyticae Praxis, An English Translation with Commentary,** ed. and trans. by Muriel Seltman and Robert Goulding, Springer, 2007
5. **Thomas Harriot, Renaissance Scientist,** ed. by John. W. Shirley, Oxford University Press, 1974
6. **Thomas Harriot, A Biography,** John W. Shirley, Oxford University Press, 1983
7. **A Discourse Concerning Algebra, English Algebra to 1685,** Jacqueline A. Stedall, Oxford University Press, 2002