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Kepler's Development of Mathematical Astronomy

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Johannes Kepler was born in Germany on December 27th, 1571. At the age of 18, he attended the University of Tübingen where he would expand his skills as both a mathematician and astronomer [1, p. 23-46]. It was at Tübingen where Kepler was converted to Copernicanism: the view that the sun is at the center of the universe and its planets rotate around it in circular orbits [3, p. 357]. Kepler's first book was published in 1596 and was titled, *Mysterium Cosmographicum* [The Cosmographic Mystery]. In it, Kepler sought to answer scientific questions relating to Copernican cosmology and attempted to make sense of them through "the mind of the Creator" [10, p. 8].

Kepler's work with the Copernican system culminated in his publishing of *Epitome Astronomiae Copernicanae* (1618) [Epitome of Copernican Astronomy]. This work, released in three sections, showed his systematic and mathematical control over the theory of astronomy. Included in the text was the revolutionary idea that astronomy was actually a branch of physics [10, p. 1-2]. Kepler writes to the reader in Book IV of his *Epitome* that:

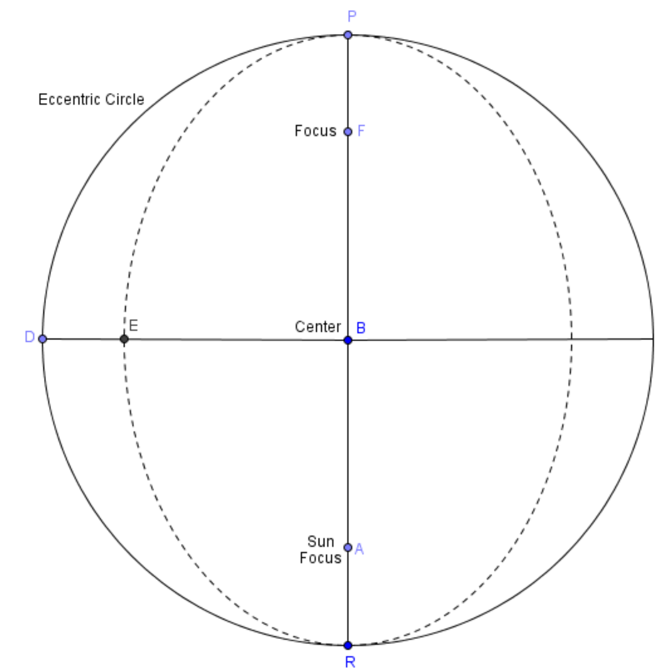
> …This Fourth Book, which airs so many new and unthought of things concerning the whole nature of the heavens, -- so that you might doubt whether you were doing a part of physics or astronomy, unless you recognized that speculative astronomy is one whole part of physics [7, p. 1].

With this framework in mind, I invite the reader to take a closer look at Kepler's work in the mathematics of astronomy, beginning with some preliminary information known during that time period.

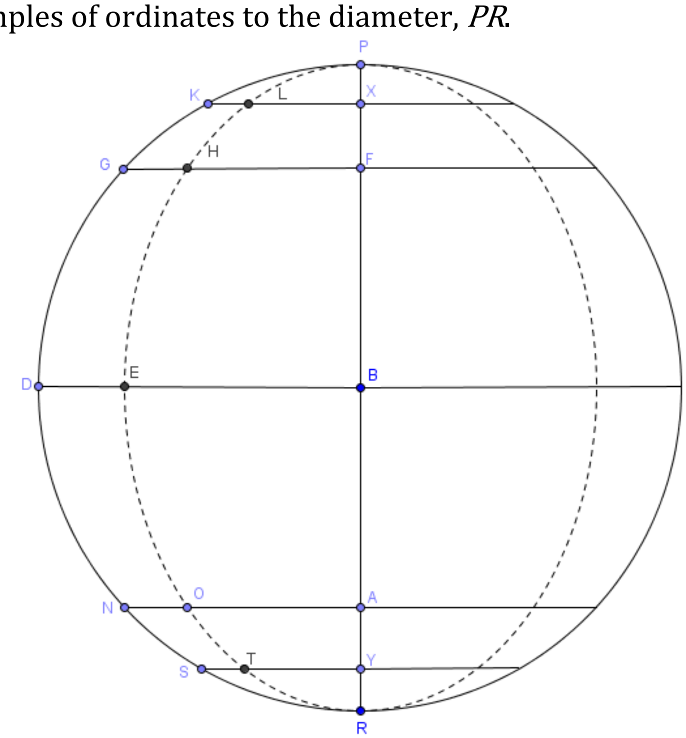
[Preliminary One:]

> It is clear from the Conics of Apollonius of Perga [circa 250-175 BCE] [6, p. 114] that the ellipse around which a circle is circumscribed, with the longer diameter of the ellipse as the common diameter, cuts all the ordinates to that diameter in the same ratio of the segments. [7, p. 226]

From this point on, Kepler uses the term "eccentric circle," to describe the circle *PDR* which has center *B* and circumscribes the ellipse *PER* [4, p. 212-213].



The ordinates of the diameter, *PR*, are the perpendicular line segments terminated by the diameter and the circle. Hence, in the picture below, lines *KX, GF, DB, NA,* and *SY* are examples of ordinates to the diameter, *PR*.



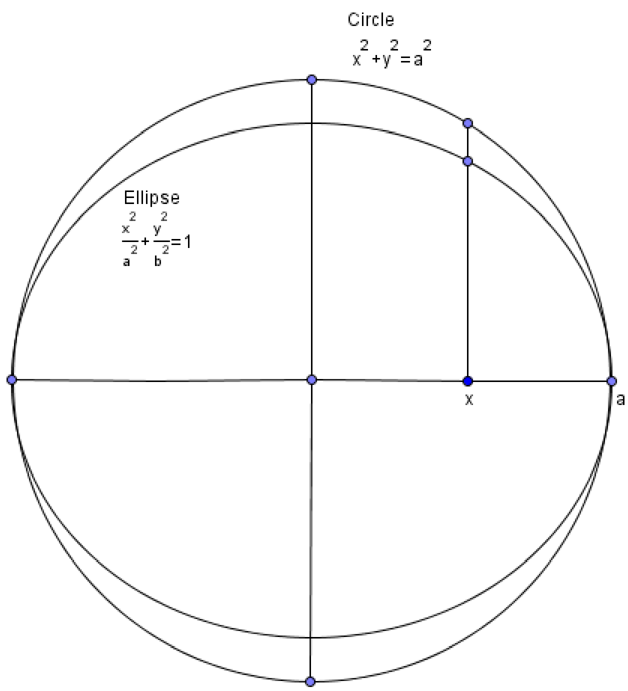
[Preliminary One, continued]

> If the lines KX, GF, DB, NA, and SY are ordinates applie[d] to PR, and if the curved line PLHEOTR is an ellipse, then necessarily as DB is to BE, so is GF to FH, so KX to XL, so NA to AO, and SY to YT [7, p. 227].

This details a specific property of ellipses and eccentric circles which states that the ratios of an ordinate to the diameter, to the section of the ordinate cut off by the ellipse, are all equal.

Meaning, $\frac{DB}{BE} = \frac{GF}{FH} = \frac{KX}{XL} = \frac{NA}{A0} = \frac{SY}{YT}$. This can be shown geometrically, but for the sake of brevity I will show a modern analytic representation of this property.

Suppose $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, an ellipse, and $x^2 + y^2 = a^2$, the circle circumscribing the ellipse, are given. Fix a point x on the diameter and then solve for y^2 in both equations:



$$\begin{cases} y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) = \frac{b^2}{a^2}(a^2 - x^2) \\ y^2 = a^2 - x^2 \end{cases}$$

So, $\frac{distance\ from\ the\ circle\ to\ the\ diameter}{distance\ from\ the\ ellipse\ to\ the\ diameter} = \sqrt{\frac{a^2 - x^2}{\frac{b^2}{a^2}(a^2 - x^2)}} = \sqrt{\frac{a^2}{b^2}} = \frac{a}{b}$.

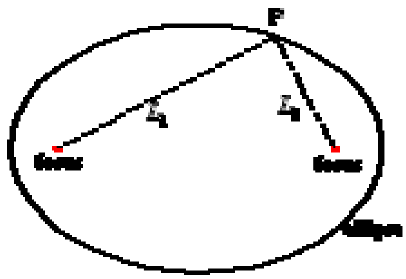
Meaning, the ratios above are all equal to the constant value $\frac{a}{b}$.

[Preliminary Two:]

> [a.]The ellipse has two points, from which it is described [drawn] as from centers; I am accustomed to call these two points the foci [7, p. 227].

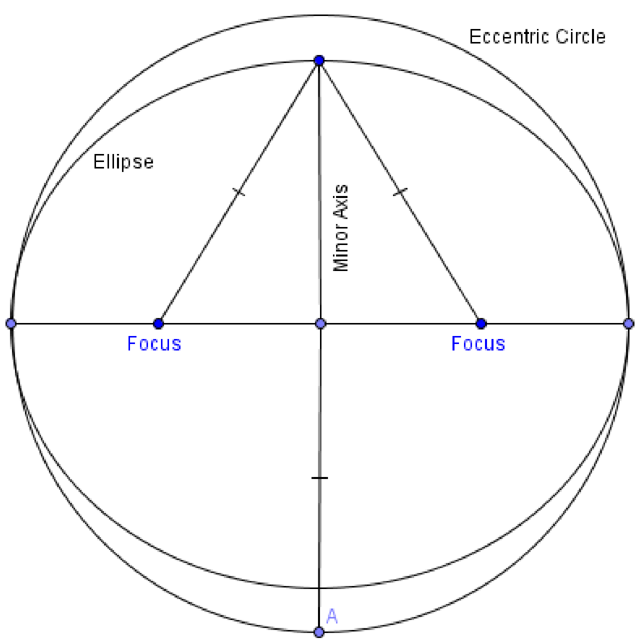
The term "foci" was first introduced by Kepler in his book *Astronomiae Pars Optica,* [The Optical Part of Astronomy] which was published earlier in 1604 [2, p. 676]. We still use this terminology today.

> [b.] Accordingly if the lines drawn from the two foci to any point on the ellipse, or even the lines drawn from one focus to the points opposite the center of the ellipse, are added together, they are always equal to the longer diameter [major axis] [7, p. 227].



The sum $L_1 + L_2$ is constant, no matter where point P is taken on the ellipse, even if P is on the major axis. This constant distance is equal to the length of the major axis of the ellipse.

> [c.] Hence when they [lines drawn from the two foci] are drawn to those points on the ellipse which are in [the] shorter diameter [minor axis] lying midway between the vertices, each of them is equal to the semidiameter [radius] of the [eccentric] circle [7, p. 227].

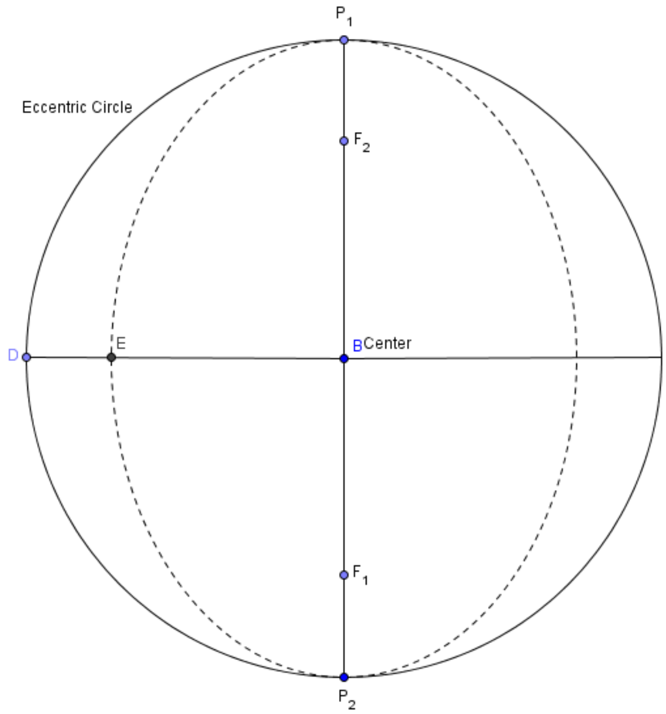


Kepler's astronomical work combined his knowledge of mathematics with the planetary and lunar observations of his time. His primary source of astronomical data was made available to him by Tycho Brahe (1546-1601). Tycho's observations were the culmination of thirty years of work and were highly sought after during that time period for their unprecedented accuracy [3, p. 356]. Kepler set out to expand the science of astronomy, using these observations as a base for his work.

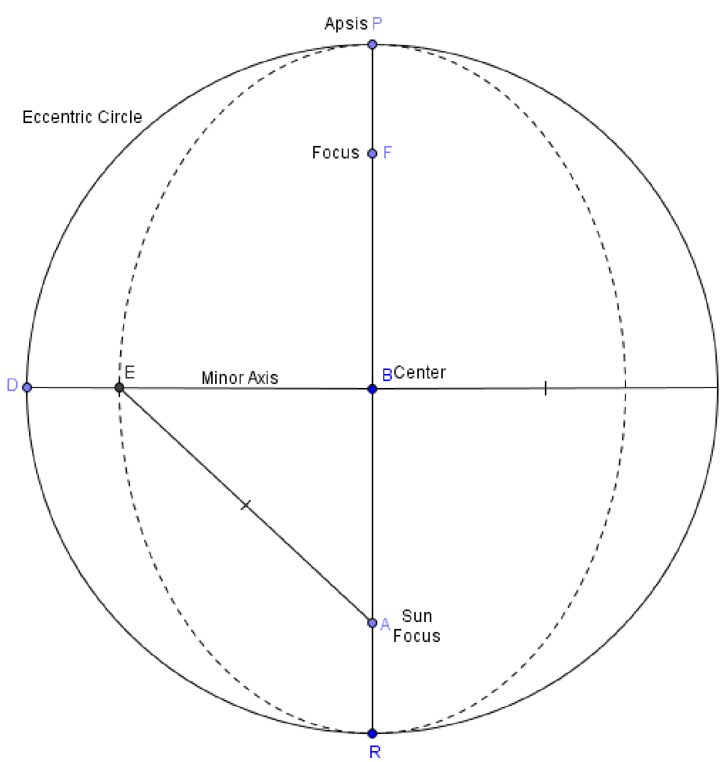
Kepler writes in Book V Part III of the *Epitome* that:

> This is applied to the planets, as follows: we have said that observations bear witness [Tycho's observations consisted of data from 1551 to 1577] [6, p. 446] that the planets are at a distance of the semidiameter of the eccentric circle [radius of the circle circumscribing the ellipse] from the sun, -- one focus of this ellipse, -- at a time when they have traversed exactly a quadrant of the orbit from apsis P [defined below] [7, p. 227]. .

This passage highlights a property of ellipses noted in Preliminary Two (c). Specifically, the distance from each focus to the point on the ellipse which also lies on the minor axis is equal to the semidiameter [radius] of the circle circumscribing the ellipse. In this passage, we are given that the planetary orbits are traced from the apsis P, which is defined as the point of greatest distance of a body from one of the foci of its elliptical orbit. Kepler uses the term "apsis P" referring exclusively to the point P which is the greatest distance from the focus indicating the sun's position, but to be clear, each ellipse has two apsis points depending on the focus in question. This is illustrated on the next page, where P_1 is the point of greatest distance [apsis] from the focus F_1. Similarly, P_2 is the apsis of the focus F_2.



The passage also says that the planet is "at a distance of the semidiameter of the eccentric circle from the sun" when it has "traversed exactly a quadrant of the orbit from apsis *P*." This means that when the planet lies on the minor axis of the ellipse, the distance from the planet to the sun $[AE]$ is equal to the radius of the circle.

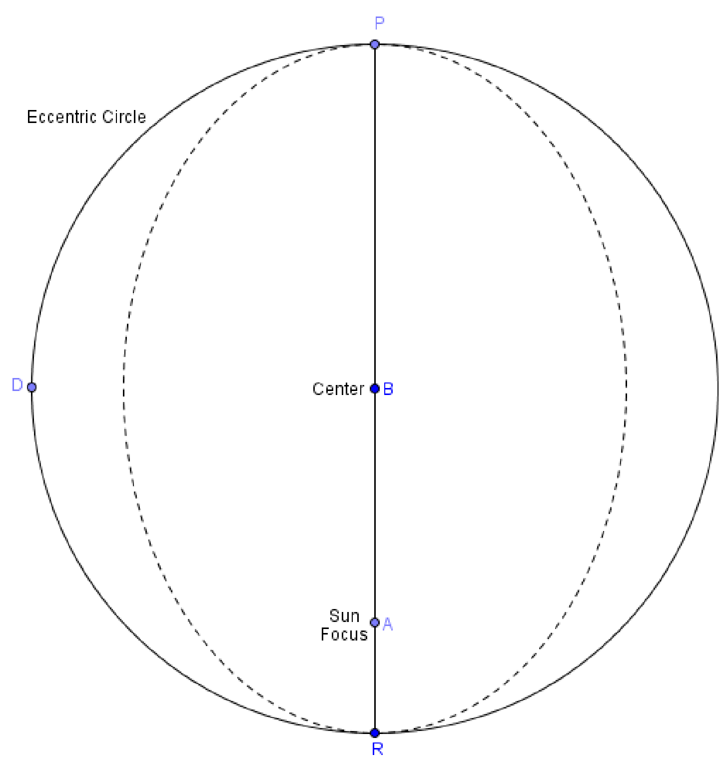


Hence, by Preliminary Two (c) we know that "observations bear witness" to the fact that the sun is one of the foci of the ellipse created by tracing the orbit of any of the planets.

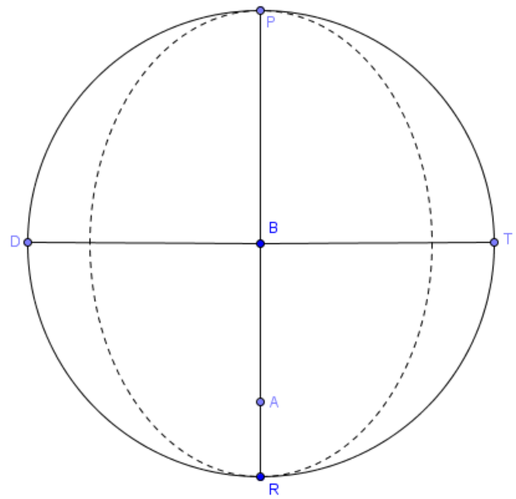
Following the previous passage, Kepler makes a series of constructions and claims that further apply properties of an ellipse to the planetary orbits being observed. From these constructions and claims, Kepler is able to develop many facts relating to planetary orbits which will eventually lead him to form his laws of planetary motion. Due to the volume of his work, I will highlight only one section from his *Epitome*, in order to provide some insight and understanding to the methods and style that Kepler employed during his investigation of the orbits of the planets.

The passage below was translated from Latin to English by Charles Glenn Wallis and is taken directly from Book V of Kepler's *Epitome* [7]. It begins with a construction, which is illustrated further on, from which a claim is made and proved. Kepler writes:

> Let there be described a new figure, namely, with center B, the circle PDR, to which the ellipse should be tangent. Let PR be the longer diameter of the ellipse [the major axis of the ellipse], and on PR let A be a focus, or the place of the sun [this follows from the preliminary above showing that Kepler's observations bear witness to the sun being one focus of the ellipse] [7, p. 228].



Now let DT be drawn through [the center] B perpendicular to PR; the shorter diameter [minor axis of the ellipse] will be on DT.



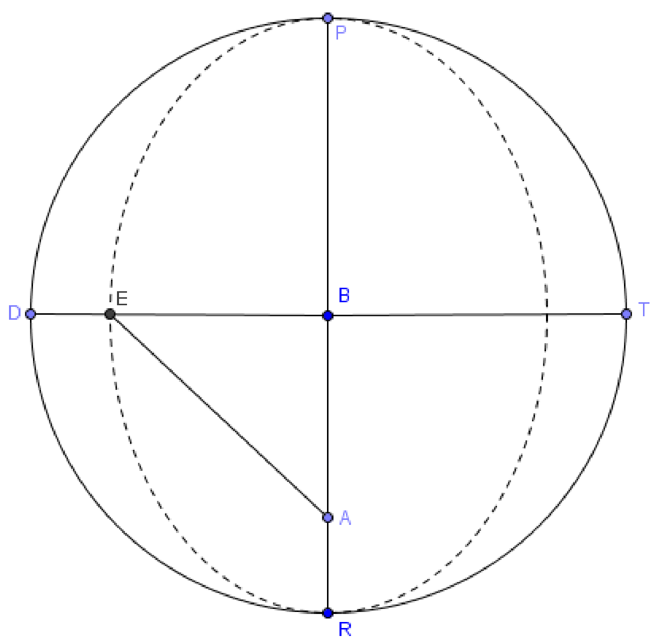
And because BA the eccentricity [the focal distance, meaning the distance from the focus to the center of the ellipse] is half of the libration [defined below] the same quantity goes to the complete quadrant.

The libration is the maximum distance a planet on the ellipse can get from a focus minus the minimum distance a planet can get from a focus. This distance is twice the focal distance.

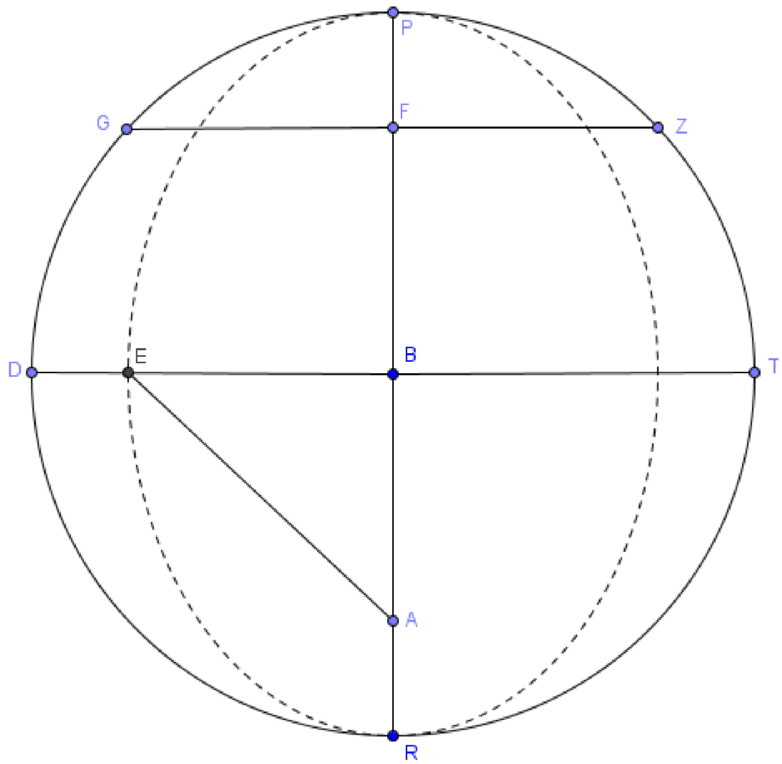
Therefore the planet falling upon line DB [midway between the two vertices] will be less distant from the sun than at P, and the difference will be equal to BA. Therefore it will have a distance equal to the magnitude PB [by Preliminary 2 (c)].

Wherefore let an interval equal to PB be extended from A to DB, and let its terminus be E [hence,$AE = PB$].

Accordingly the orbit of the planet will cut DB at E.

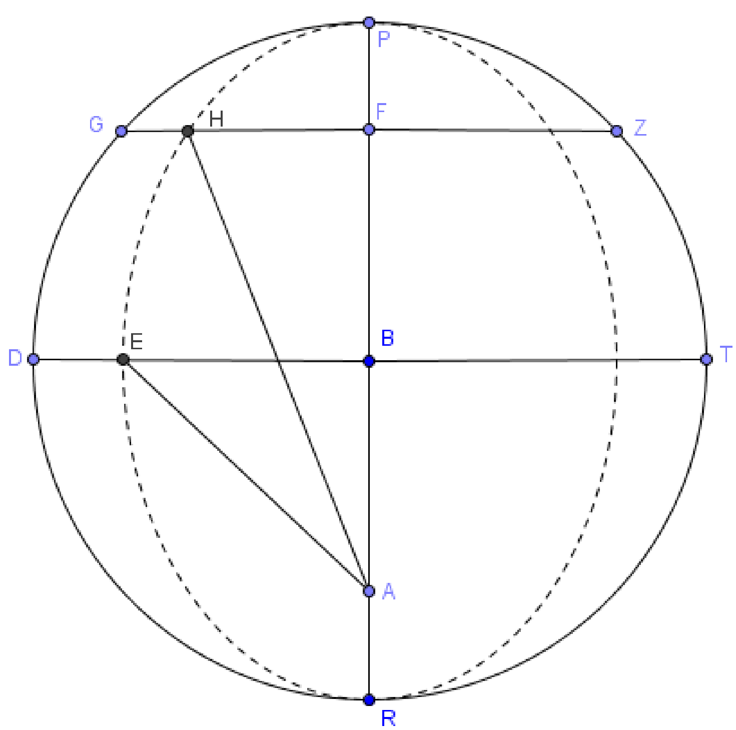


Again let there be taken PG an [arbitrary] arc of the circle, and GF its … ordinate [to the diameter].



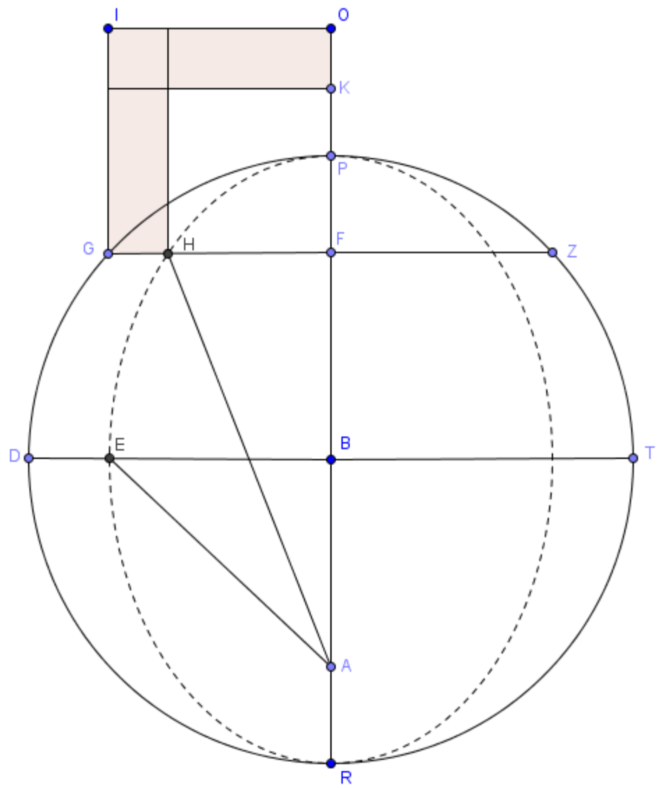
(⋆) Accordingly make BP [the radius] be to PF as BA half of the libration [the focal distance] is to the part belonging to PG [HF-PF]. $\left[\text{Meaning,} \frac{BP}{PF} = \frac{BA}{HF-PF}\right]$.

And when that [part] has been subtracted from AP [the difference HF-PF is subtracted from the distance of the focus A to its apsis P], let the remainder be extended from A to GF, and let its terminus fall upon H.

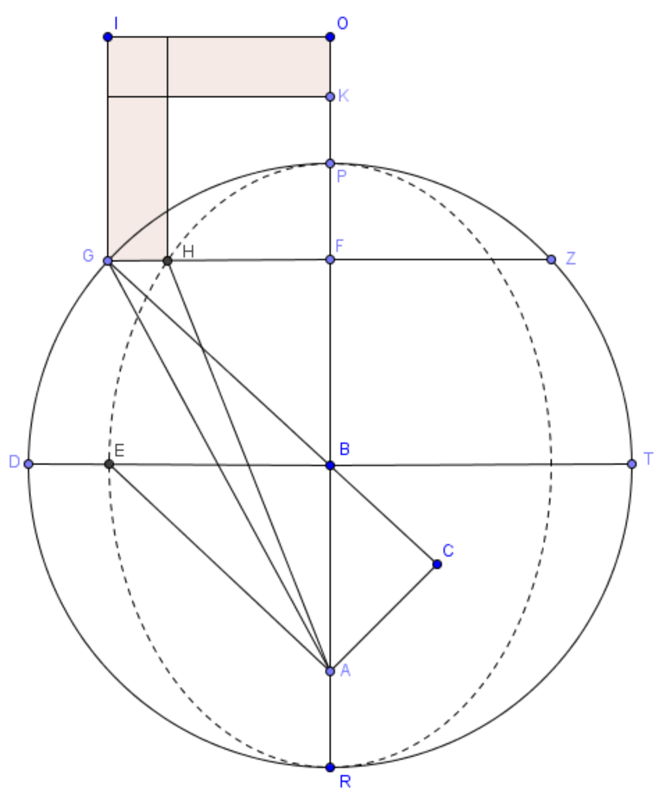


For let square GIOF be described on GF, but square HK on HF, -- so that gnomon HIK is made.

The gnomon *HIK* is the shaded region below that is created by removing the square on *HF* from the square on *GF*.



Then let GA and GB be joined and let the perpendicular AC be drawn to GB continued.



[Claim:] I say in the first place that the square on AC is equal to the gnomon HIK.

[Proof:]

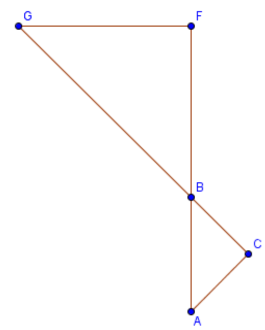
For because, as BP is to PF, so is BA to the difference between lines AP and AH $\left[\frac{BP}{PF} = \frac{BA}{AP-AH}, \text{by } (\star)\right]$. Wherefore also as BP is to BF so is BA to the excess of AH over BP $\left[\frac{BP}{BF} = \frac{BA}{AH-BP}\right]$.

The equality $\frac{BP}{BF} = \frac{BA}{AH-BP}$ can be easily found using $(\star)$ as follows:

Invert $\frac{BP}{PF} = \frac{BA}{AP-AH}$ to get $\frac{PF}{BP} = \frac{AP-AH}{BA}$. Substituting $PF = BP - BF$ and $AP = BA + BP$ into $\frac{PF}{BP} = \frac{AP-AH}{BA}$ yields $\frac{BP-BF}{BP} = \frac{BA+BP-AH}{BA}$. Thus, $1 - \frac{BF}{BP} = 1 - \frac{AH-BP}{BA}$. Simplifying gives us $\frac{BF}{BP} = \frac{AH-BP}{BA}$. By inverting this equality we obtain $\frac{BP}{BF} = \frac{BA}{AH-BP}$, as desired.

But too as BP or GB [$BP = GB$ since they are both radii of the eccentric circle] is to BF, so is BA to BC, because the right triangles GFB and ACB have their vertical angles GBF and ABC equal [congruent].

So, $\frac{GB}{BF} = \frac{BP}{BF} = \frac{BA}{BC}$.



Therefore BC is equal to the portion whereby AH exceeds BP.

This means $BC = AH - BP$, since $\frac{BA}{BC} = \frac{BP}{BF} = \frac{BA}{AH-BP}$.

But GC also exceeds BP. i.e., BG, by this same portion BC. Wherefore

GC and HA are equal.

Here Kepler adds that $GC - BP = BC$. We can substitute $AH - BP$ in for BC and get $GC - BP = AH - BP$. Hence, $GC = AH$. This fact will be useful later on in showing that gnomon HIK is equal in area to the square on AC.

But if the square on the straight line GC and the square on the perpendicular AC are added together, they are equal to the square on the straight line GA [by the Pythagorean Theorem, $GC^2 + AC^2 = GA^2$, in right triangle GCA]. But, on the other hand, if the square on AF and the square on FG are added together, they are equal to the square on the same GA [$AF^2 + FG^2 = GA^2$, in right triangle AFG]. Therefore the sum of the two squares on GF and FA is equal to the sum of the squares on GC and CA [by substitution, $AF^2 + FG^2 = GC^2 + AC^2$].

Therefore if the square on GC is subtracted from this second sum, the square AC is left; and if from the first sum the square on line AH, -- which is equal to GC --, i.e., the two squares, on AF and on FH, viz., square HK, are subtracted, the gnomon HIK is left.

Kepler now uses the previously derived fact that $GC = AH$. He subtracts the square of GC on the right hand side of $AF^2 + FG^2 = GC^2 + AC^2$ and he subtracts the square of AH on the left hand side. This gives us $AF^2 + FG^2 - AH^2 = AC^2$, or as Kepler says, "the square on AC is left." When Kepler says "the two squares, on *AF* and on *FH*, viz., square *HK*" he is referring to the right triangle *HFA*. From the Pythagorean Theorem, $HA^2 = AF^2 + FH^2$ so $AF^2 - AH^2 = -FH^2$. Substituting this into the left hand side of $AF^2 + FG^2 - AH^2 = AC^2$ gives us $FG^2 - FH^2 = AC^2$. Thus, the area of the gnomon *HIK* is equal to the square on AC■

The proof of this claim gives us a taste of the style used by Kepler in his *Epitome*. Following this proof, Kepler proposes and justifies similar statements relating to other properties of the eccentric circle and the ellipse. These properties, along with the astronomical observations available at the time, led Kepler to many unique physical theories on the orbits of the planets; most notably his three laws of planetary motion [9, p. 7]. Kepler's *Epitome* was avidly read by his contemporaries and successors primarily for its astronomical content, though initially it was received with mixed reviews [3, p. 359]. A sample of the criticism given at that time came from a Professor of Mathematics at Danzig, named Peter Crüger. In 1622, Crüger wrote:

> I have received the fourth book of Kepler's astronomy… The Poet says that to read a thing ten times is pleasing. But this work I do not yet understand after reading it a hundred times. The author seems as usual, to obscure the matter deliberately. However, I will study all these things later at leisure with my whole strength, though I do not see what use this will be. These theories are based upon uncertain foundations and mere guesswork [9, p. 8].

Other critics offered similar views until 1627, when Kepler published *Tabulae Rudolphinae* [the Rudolphine Tables]. The *Rudolphine Tables* was comprised of explanations of improved methods of observation and the numerical tables that accompanied such work. In his untitled review of *Tabulae Rudolphinae,* Gingerich writes:

> When Kepler became an astronomer in the closing years of the sixteenth century, he found a science which planetary predictions typically erred by several degrees on the sky; the legacy of the Rudolphine tables was a prediction scheme nearly fifty times more accurate. For Kepler these tables were the proof of the pudding, the substantiation of his laws of planetary motion. He called them "my chief astronomical work" [5, p. 125].

The accuracy of Kepler's tables cleared up many of the issues held against his *Epitome* and laws of planetary motion. This is evidenced by a follow-up letter from Peter Crüger, written in 1629 after receiving Kepler's tables:

> I am wholly occupied with trying to understand the foundations upon which the Rudolphine rules and tables are based, and I am using for this purpose the Epitome of Astronomy previously published by Kepler as an introduction to the tables. This epitome which previously I had read so many times and so little understood and so many times thrown aside, I now take up again and study with rather more success seeing that it was intended for use with the tables and is itself clarified by them…I am no longer repelled by the elliptical form of the planetary orbits; Kepler's proofs, in his *Astronomia Nova* have convinced me [9, p. 8].

Kepler's unique vision and treatment of astronomy provided a firm base for which later minds could expand the modern version of the science. His work was accepted and remained one of the most reliable sources of astronomy for many decades preceding the great *Principia*, by Isaac Newton [10, p. 1]. Kepler wrote in his poetic epitaph,

> *Mensus eram coelos, nunc terrae metior umbras*
> *Mens coelestis erat, corporis umbra iacet.*
>
> I measured the skies, now the shadows I measure
> Skybound was the mind, earthbound the body rests [8, p. 427].

He passed away on November 15th, 1630, but his work and legacy is still remembered to this day.

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