The Mathematical Optics of Sir William Rowan Hamilton: Conical Refraction and Quaternions

Nathan McLaughlin
University of Montana

Home Address:

410 West River Rd.
Hamilton, MT 59840

Email:

nathan.mclaughlin@umontana.edu

The Irish mathematician, astronomer, and physicist Sir William Rowan Hamilton made an enormous number of contributions to his fields. As a result, these fields have immortalized Hamilton in the numerous equations and concepts which bear his name. In 1833 he published a paper describing a characteristic function determining the behavior of rays. When Hamilton applied this function to Fresnel's equations for the wave surface of biaxal crystals he was able to predict the phenomenon of conical refraction. The mathematical discovery for which Hamilton is perhaps best known came in 1843 when he described quaternions. Twenty nine years after the discovery of conical refraction and nineteen years after that of quaternions, Hamilton wrote a paper which combined the two ideas to give a new quaternion form of the equations for a wave surface for biaxal crystals. This paper was entitled *On Some Quaternion Equations Connected with Fresnel's Wave Surface for Biaxal Crystals.* It did not present any new physical or mathematical concepts, but, because it combines some of Hamilton's greatest ideas, it provides an insightful summary of the highlights of his work.

The Life of Hamilton

William Rowan Hamilton's life began in Dublin, Ireland in 1805. When only three years old Hamilton went to live with his uncle who also became Hamilton's tutor in preparation for university study. Hamilton's brilliance was evident early on, as was perhaps most obvious in his familiarity with thirteen languages by the time he was thirteen. In addition, he read extensively in the subjects of astronomy, religion, mathematics, literature, and geography. Hamilton began study at Trinity College of Dublin in 1823 and received an unprecedented number of awards during his three years there. He then became the Royal Astronomer of Ireland and professor of Astronomy at the University of Dublin. Hamilton's research in mathematical optics began when he was only seventeen. This research resulted in his famous "Theory of Systems of Rays", published in 1828. Hamilton sought to "reduce optics to a mathematical science in terms of his Characteristic Function"(Crowe, 21). Once he had achieved this, Hamilton went on to apply these methods to dynamics. Unlike many mathematicians, Hamilton's work brought him considerable fame. This was particularly true when, in 1842 and still less than thirty years old, he theorized the phenomena of internal and external conical refraction. When this prediction was verified experimentally by Humphrey Lloyd it excited scientists across Europe and was called "perhaps the most remarkable prediction that has ever been made"(Crowe, 21). Hamilton's fame continued to grow, and in 1835 he was knighted and received a medal from the Royal Society. Two years later he was elected president of the Royal Irish Academy. He continued his mathematical work until 1865, the year of his death.

Given the astounding number of accomplishments Hamilton achieved, it is surprising that some people have considered his life a tragedy. The list of Hamilton's contributions to science is almost endless and includes his Hamiltonian mechanics (which became vital to the fields of quantum mechanics and electromagnetism), the invention of Icosian Calculus, Hamilton's principle, the Hamilton-Jacobi equation, Hamiltonian groups, and the discovery of quaternions. It is the last of these contributions, quaternions, which has caused some people to view Hamilton's life as tragic. Hamilton discovered quaternions in 1843 and spent the rest of his life devoted to them. He strongly believed that they represented the future of mathematics, but many mathematicians have viewed it as "one of many comparable mathematical systems and ... it offers little value for application"(Crowe, 18). This point of view has been argued against by many important scientists (Whittaker, Birkhoff, Dirac, et al.), who felt that quaternions would become more widely used in the future. Quaternions have seen some increased popularity due to their applications to computer science, but they have become nearly extinct in other fields. It is not possible to determine whether quaternions will someday become as important as Hamilton expected, nor to judge whether he wasted the last twenty-two years of his life on a delusional"belief that quaternions held the the key to the mathematics of the physical universe"(Bell,404). Despite this, the amount of effort spent in seeking out and developing quaternions along with the simplifications they made to systems such as the biaxal crystal make it easy to see why Hamilton would be convinced that quaternions were the future.

The Biaxal Crystal and Conical Refraction

In 1832 Hamilton began a study of the wave surface theorized by Augustin-Jean Fresnel which describes light propagation from within a biaxal Crystal. He based this study on his general theory developed in the *Third Supplement to an Essay on the Theory of Systems of Rays* . The result of this study was the prediction that the geometry of such a crystal should give rise to two previously unobserved phenomenon: internal and external conical refraction. At Hamilton's request, Humphrey Lloyd experimentally verified the prediction using a piece of arragonite and published his results in 1833. This was arguably the first time that the mathematical analysis of a physical phenomenon preceded its experimental verification. In addition, the discovery greatly advanced the development of optics by providing a strong argument in support of the theory that light has a transverse wave nature.

Hamilton based his work on the idea that a light wave passing through a medium causes a small displacement of the molecules within that medium. The most modern and physically accurate technique for describing the optical properties of a medium such as a biaxal crystal uses the dielectric tensor ϵ and the tensor of magnetic permeability μ (see, for instance, Born and Wolf's *Principles of Optics*). However, since $v = c/\sqrt{\mu\epsilon}$ (where v is light's velocity in some medium and c is its velocity in a vacuum), perhaps a more intuitive approach is to focus solely on the velocity of light through the crystal. In order to investigate the behavior of light in a crystalline material, it is useful to employ geometrical constructions which determine the propagation velocities and vibration directions of the wave. Such constructs are called wave surfaces. Since crystals are effectively rigid bodies, motion within them depends only on three numbers (Thornton and Marion, 447). The simplest shape that such a body can posses is an ellipsoid, so the motion of any rigid body can be represented by the motion of an equivalent ellipsoid. When this ellipsoid is based on the refractive index of the crystal it is called "the index ellipsoid". Its equation is

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1,$$

where A is inversely proportional to the velocity of wave propagation in the x direction, B to that in the y direction, and C in the z direction (to be precise, $A = (c/\sqrt{\mu})v_x^{-1}$, $B = (c/\sqrt{\mu})v_y^{-1}$ and $C = (c/\sqrt{\mu})v_z^{-1}$). This is called the index ellipsoid because the refractive index in the x direction is simply $\sqrt{\mu}A$ and similarly for the y and z directions. Now let $\vec{k}$ be a unit vector normal to a wave. The propagation of this wave within a crystal gives rise to vibrations in two directions perpendicular to $\vec{k}$. When a plane is drawn through the origin of the ellipsoid and perpendicular to $\vec{k}$, an ellipse is formed by the intersection of this plane and the ellipsoid (Figure 1).

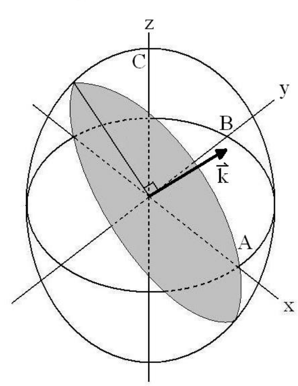

Figure 1: Index Ellipsoid. $\vec{k}$ is normal to a wave propagating through the crystal.

The semiaxes of this ellipse point in the directions of vibration and the lengths of these axes are proportional to the velocity. Since this ellipse is perpendicular to $\vec{k}$ and constrained by the ellipsoid, it must satisfy the equations

$$xk_x + yk_y + zk_z = 0 \qquad \text{and} \qquad \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1,$$

These properties allow for the derivation of Fresnel's equation of wave normals, which is (Born and Wolf, 806):

$$k_x^2(v_p^2 - v_y^2)(v_p^2 - v_z^2) + k_y^2(v_p^2 - v_z^2)(v_p^2 - v_x^2) + k_z^2(v_p^2 - v_x^2)(v_p^2 - v_y^2) = 0 \tag{1}$$

Where k_x is the x component of $\vec{k}$, so that (1) is the equation of a quadric surface in "k space". The property which distinguishes biaxal crystals from other forms is that they have three different refractive indices and therefore $A \neq B \neq C$. To see the consequences of this property assume $A < B < C$, which implies that $v_x > v_y > v_z$. Considering just the k_y, k_z plane by setting $k_x = 0$ in (1) results in two solutions. The first is simply $v_p^2 = v_x^2$ and the other is

$$k_y^2(v_p^2 - v_z^2)(v_p^2 - v_x^2) = -k_z^2(v_p^2 - v_x^2)(v_p^2 - v_y^2)$$

$$\Rightarrow \quad k_y^2(v_p^2 - v_z^2) = k_z^2(v_y^2 - v_p^2)$$

$$\Rightarrow \quad v_p^2(k_y^2 + k_z^2) = v_z^2 k_y^2 + v_y^2 k_z^2,$$

but $k_y^2 + k_z^2 = |\vec{k}|$ when $k_x = 0$ and being a unit vector $|\vec{k}| = 1$, so

$$v_p^2 = v_z^2 k_y^2 + v_y^2 k_z^2$$

Since v_p is a constant it is permissible to set $v_p k_y = y$ and $v_p k_z = z$, which makes the two solutions to Fresnel's wave equation in the yz plane

$$v_x^2 = v_p^2(1) = v_p^2(k_y^2 + k_z^2) = v_p^2 k_y^2 + v_p^2 k_z^2 = y^2 + z^2$$

$$\Rightarrow y^2 + z^2 = v_x^2 \tag{2}$$

and

$$v_p^2 = v_z^2 k_y^2 + v_y^2 k_z^2 \Rightarrow (v_p^2)^2 = v_z^2 v_p^2 k_y^2 + v_y^2 v_p^2 k_z^2 = v_z^2 y^2 + v^2 z^2$$

$$\Rightarrow (y^2 + z^2)^2 = v_z^2 y^2 + v_y^2 z^2 \tag{3}$$

Notice that (2) is the equation of a circle, while (3) is the equation of an oval. The other two planes give similar results (Figure 2). Extending this to three dimensions creates the wave surface as seen in Figure 3.

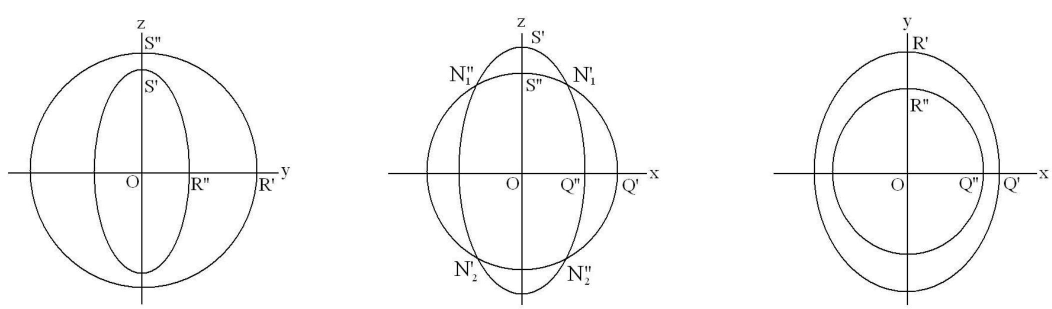

Figure 2: The projections of Fresnel's wave equations onto the three planes.

Two lines can be drawn through the origin to form $N_1'ON_2'$ and $N_1''ON_2''$. A plane perpendicular to either of these lines forms two concentric circles when it intersects the wave surface, which indicates that the surface is symmetric about these lines. These lines are called the optic axes of the crystal and the fact that there

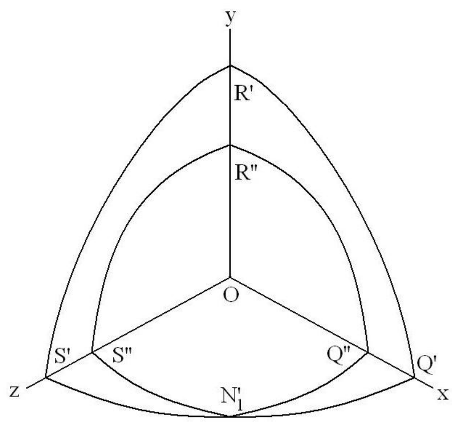

Figure 3: An octant of the wave surface of a biaxal crystal.

are two such axes in biaxal crystals why those crystals are so named. The utility of the wave surface is that it indicates a symmetry in light wave velocity about the optic axes.

Consider a wavefront (WF) incident on a rectangular biaxal crystal (Figure 4). If the angle of incidence of WF is such that the refracted wave front $ABCD$ is normal to an optic axis, the ray QR can be refracted along any direction lying on the surface of a cone. This occurs because light follows the path of least time (Fermat's principle; Pedrotti, 20) which will be the path with greatest velocity. As seen, there are an infinite number of points with the same velocity lying on the edge of a circle for each plane perpendicular to the optic axis, so light can take any path on the surface of a cone. When the rays emerge from the crystal, they form a cylinder. This is called internal conical refraction. External conical refraction is similar, but the source of the light is within the crystal and light is refracted into a hollow cone of rays when it emerges from the crystal.

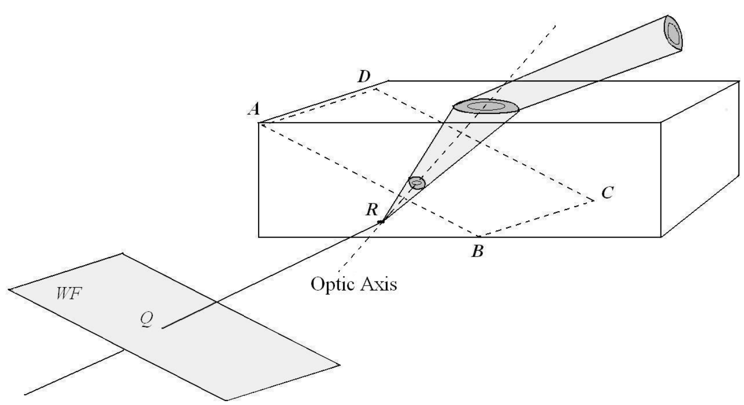

Figure 4: Internal Conical Refraction.

It is amazing that Hamilton was able to predict this phenomenon mathematically. His general approach to optics was formulate a characteristic equation (V) for the system being investigated. This equation was required to satisfy the following: light traveling from an object at $(x^{'}, y^{'}, z^{'})$ to an eye at (x, y, z) follows a path determined by the direction cosines α, β, γ and $\alpha^{'}, \beta^{'}, \gamma^{'}$ which can be determined by

$$\alpha = \frac{\partial V}{\partial x}, \quad \beta = \frac{\partial V}{\partial y} \quad \gamma = \frac{\partial V}{\partial z}$$

$$\alpha^{'} = \frac{\partial V}{\partial x'}, \quad \beta^{'} = \frac{\partial V}{\partial y'} \quad \gamma^{'} = \frac{\partial V}{\partial z'}$$

This technique proved to be very successful, so much so that Hamilton claimed mathematical optics could be divided "into two principle parts: one part proposing to find in every particular case the form of the characteristic function V, and the other part proposing to use it"(Hamilton, *On Some Results of the View of a Characteristic Function in Optics*). Hamilton also often used a quantity called the "slowness". Technically, slowness is the gradient of the action function, but for the following analysis it is only important to know that the slowness vector is perpendicular to all spatial vectors which are tangent to the wave surface (that is, it points in the direction of propagation of the wave) and that its length increases as the light's velocity decreases. Hamilton observed that $\frac{\partial V}{\partial x}$ represents the slowness of light in the x direction. Furthermore, when the normal to the refracting surface is taken to be the z-axis, then

$$\Delta \frac{\partial V}{\partial x} = 0 \qquad \text{and} \qquad \Delta \frac{\partial V}{\partial y} = 0 \tag{4}$$

This implies that the component of normal slowness is not altered by refraction in a direction parallel to the surface. This can easily be demonstrated for ordinary refraction (in a material with a single index of refraction). In Figure 5, light is incident on an object at angle θ_1. It then refracts at angle θ_2. Equation (4) says that the velocity of the ray in the x and y directions does not change upon refraction. To simplify things, consider incident light with no y-component of velocity. By letting $L1$ and $L2$ be lengths of the incident and refracted rays and making them proportional to n_1 and n_2 respectively, it follows that

$$\frac{L1_x}{time} = \frac{L2_x}{time} = V_x \quad \Rightarrow \quad L1_x = L2_x,$$

where $L1_x$ and $L2_x$ are the x-components of $L1$ and $L2$, given by

$$L1_x = L1 \sin\theta_1 \quad \text{and} \quad L2_x = L2 \sin\theta_2$$

This means that $L1 \sin\theta_1 = L2 \sin\theta_2$, or equivalently $n_1 \sin\theta_1 = n_2 \sin\theta_2$, which is simply Snell's Law.

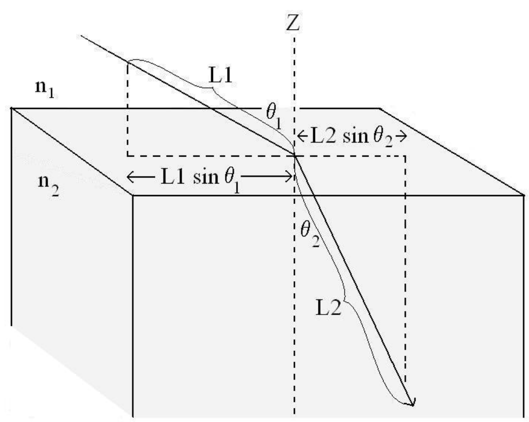

Figure 5: n_1 and n_2 represent the indices of refraction of the two objects.

In applying this method to extraordinary refraction (as is present in biaxal crystals), construction of the slowness requirement involves, instead of L2 above, a variable length due to the multiple refractive indices. Hamilton had noticed that when he constructed the locus of points in all directions of the slowness for ordinary refraction a sphere was produced, but for extraordinary refraction, the variable length resulted in a double surface (called a surface of components of normal slowness). Armed with this realization and the principles Fresnel had already determined for biaxal crystals, Hamilton proceeded to apply his general method to those crystals. The surface of components which resulted had what Hamilton called "cusps" with the properties previously given to the optic axes. He immediately concluded that if a ray of light was incident on the crystal at a cusp it would produce "no unique refracted ray,... but an infinite number of refracted rays, namely, all the perpendiculars which can be let fall from the point of incidence on the tangent cone at the cusp"(*On Some Results of the View of a Characteristic Function in Optics*). This prediction offered a test for Fresnel's principles because it seemed to contradict previous observations. Given the importance of this discovery, it is not surprising that Hamilton returned to the biaxal crystal after his development of quaternions.

Quaternions

Hamilton hoped to extend the view of complex numbers as points on a plane to three dimensional space. In *Theory of Conjugate Functions, or Algebraic Couples* (which was written in 1833 but published as part of an 1837 essay), Hamilton demonstrated that "couples", that is, coordinates (a, b) of a point on a plane are equivalent to complex numbers when written in the form $a + bi$. At the end of this essay he declared his intent to develop a similar system for three dimensions which he termed a "Theory of Triplets"(Hamilton, 422). One source of motivation for this search was that points in space or "triples" could only be added and subtracted at the time, so an active area of study involved discovering the laws of multiplication. The solution occurred to Hamilton during his now famous walk along the Royal Canal in Ireland and he was so excited by it that he carved it into the Brougham Bridge. In essence, he realized that he already knew how to multiply quadruples, so he just needed a forth coordinate, as defined by his 1843 carving:

$$i^2 = j^2 = k^2 = ijk = -1$$

Quaternions became the first significant number system which did not obey the laws of ordinary arithmetic, and they were widely used. Their application to three dimensional rotations proved extremely useful in physics, since vectors had not yet been developed. They could do things that were otherwise impossible at the time and, as such, it is not surprising that Hamilton saw quaternions as the future.

As a set, the quaternions are a four dimensional vector space over the real numbers. As such, every element of the set of quaternions can be written as a linear combination of the base elements of $\mathtt{R}^4$, which are usually denoted $(1, i, j, k)$. Therefore, each quaternion is a hypercomplex number having the form $a1 + bi + cj + dk$, where $a, b, c,$ and d are real numbers and $i, j,$ and k are unit vectors along the $x, y,$ and z axes. The laws for $i, j,$ and$k$ can be easily derived from the above formula:

$$i^2 = j^2 = k^2 = ijk = -1 \;\Rightarrow\; -1 = ijk \;\Rightarrow\; -k = ijkk \;\Rightarrow\; -k = ij(-1) \;\Rightarrow\; k = ij$$

The other laws can be arrived at in the same way. They are

$$jk = i \qquad ki = j \qquad ji = -k \qquad kj = -i \qquad ik = -j$$

Using these laws, the Hamilton product can be defined,

$$(a_1 + b_1 i + c_1 j + d_1 k)(a_2 + b_2 i + c_2 j + d_2 k) = a_1a_2 + a_1b_2 i + a_1c_2 j + a_1d_2 k + b_1a_2 i + b_1b_2 i^2 + b_1c_2 ij + b_1d_2 ik +$$

$$c_1a_2 j + c_1b_2 ji + c_1c_2 j^2 + c_1d_2 jk + d_1a_2 k + d_1b_2 ki + d_1c_2 kj + d_1d_2 k^2$$

$$= a_1a_2 + a_1b_2 i + a_1c_2 j + a_1d_2 k + b_1a_2 i - b_1b_2 + b_1c_2 k + b_1c_2 j + c_1a_2 j - c_1b_2 k - c_1c_2 + c_1d_2 i + d_1a_2 k + d_1b_2 j - d_1c_2 i - d_1d_2$$

$$= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)i + (a_1c_2 + b_1c_2 + c_1a_2 + d_1b_2)j + (a_1d_2 + b_1c_2 - c_1b_2 + d_1a_2)k$$

Despite what the complex equation above may suggest, quaternions are very easy to work with and the calculation of the product given above is rarely necessary. One useful simplification comes the quaternions being multiplied are conjugates. The quaternion conjugate is analogous to the complex conjugate in that the conjugate of $q = a + bi + cj + dk$ is $q^* = a - bi - cj - dk$. It can be seen from the definition of multiplication that

$$qq^* = q^*q = a^2 + b^2 + c^2 + d^2$$

is always a real, non-negative number. In this system, a quaternion is defined as the sum of a scalar and a vector so that every quaternion can be separated into its scalar and vector parts using the notation

$$\mathrm{S}q = a, \quad \mathrm{V}q = bi + cj + dk, \quad \text{and} \quad q = \mathrm{S}q + \mathrm{V}q$$

Therefore, a vector is a quaternion with no scalar part and a scalar is a quaternion with no vector part. The scalar part of the product of two vectors $\mathrm{S}\alpha\beta$ is defined as the product of the length of α and the projection of β onto α. This is equivalent to the dot product in vector analysis. Similarly, $\mathrm{V}\alpha\beta$ is the cross product $\alpha \times \beta$. Thus, the scalar part of a vector product has commutative and distributive properties:

$$\mathrm{S}\alpha\beta = \mathrm{S}\beta\alpha \quad \text{and} \quad \mathrm{S}\alpha(\beta + \gamma) = \mathrm{S}\alpha\beta + \mathrm{S}\alpha\gamma$$

The vector part of these products does not have these properties, but $\mathrm{V}\alpha\beta = -\mathrm{V}\beta\alpha$. The product of two vectors is given by $\alpha\beta = \mathrm{S}\alpha\beta + \mathrm{V}\alpha\beta$ from which follows $\beta\alpha = \mathrm{S}\alpha\beta - \mathrm{V}\alpha\beta$. There are also useful vector functions. A linear vector function ϕ is a function that satisfies

$$\phi(\alpha + \beta) = \phi\alpha + \phi\beta, \quad \mathrm{S}\phi\alpha = 0, \quad \text{and} \quad \mathrm{S}\phi\beta = 0$$

for all vectors α and β. The usefulness of such functions will be seen shortly. There are many other useful concepts in quaternion mathematics. The formulas included here are restricted to those pertinent to this discussion. For a more complete list see Charles Joly's, *Manual of Quaternions* , from which most of the following were obtained, or Hamilton's, *Seventh Lecture on Quaternions*.

- Some useful functions in quaternion mathematics:
  - The Tensor of a vector α is its magnitude and is denoted $\mathrm{T}\alpha$
  - The vector reciprocal α^{-1} has the opposite direction as α and the reciprocal magnitude, that is,

    $$\mathrm{S}\alpha^{-1} = \mathrm{T}\alpha^{-1} = \frac{1}{\mathrm{T}\alpha} \tag{5}$$

  - Relations concerning α^2:

    $$\mathrm{V}\alpha\alpha = 0, \quad \mathrm{S}\alpha\alpha = -(\mathrm{T}\alpha)^2, \quad \text{and} \quad \alpha\alpha = \mathrm{S}\alpha\alpha + \mathrm{V}\alpha\alpha = -(\mathrm{T}\alpha)^2 \tag{6}$$

  - If α is perpendicular to β, then

    $$\mathrm{S}\alpha\beta = 0. \tag{7}$$

    Also, if $\mathrm{V}\alpha\beta = \gamma$, γ is perpendicular to both α and β. Perpendicular vectors are sometimes referred to as rectangular.
  - For an arbitrary vector ρ, the linear vector function ϕ can be chosen such that

    $$\mathrm{S}\rho\phi\rho = \text{constant} \tag{8}$$

    represents the equation of any central surface of second order (i.e. any quadric surface).
  - For a constant scalar x, $\phi(x\alpha) = x\phi\alpha$.
  - If $\sigma = \phi\rho$, then a transformation can be made from the vector ρ to σ which follows the rules

    $$\mathrm{V}\alpha\beta \quad \text{becomes} \quad \mathrm{V}\phi\alpha\phi\beta \quad \text{and} \quad \mathrm{S}\alpha\beta \quad \text{becomes} \quad \mathrm{S}\phi\alpha\phi\beta$$

    so that, as an example, the equation of a plane $\mathrm{S}(\rho - \alpha)\beta = 0$ becomes $\mathrm{S}(\sigma - \phi\alpha)\phi\beta = 0$.

As mentioned, once Hamilton had discovered quaternions they became the central focus of the remainder of his life. As such, he often would revisit his previous work with the hope of improving it with his new discovery.

Quaternion Equations for Fresnel's Wave Surface

The power of Hamilton's quaternion system reveals itself in the drastic simplifications it makes to notation. An example of this immediately presents itself in Hamilton's paper, *On Some Quaternion Equations Connected with Fresnel's Wave Surface for Biaxal Crystals*, which begins with the simple quaternion equation for an ellipsoid,

$$S\rho\phi\rho = 1. \tag{9}$$

Since quaternion equations have become somewhat obscure, it may not be obvious that (9) is the equation of an ellipsoid. To demonstrate this, the following steps transform (9) into Cartesian form. The locus of points created by rotating the vector $\rho = x\hat{x} + y\hat{y} + z\hat{z}$ in all possible directions forms a shell. Define the vector function ϕ by

$$\phi = \alpha^{-1}\mathrm{S}\alpha^{-1} + \beta^{-1}\mathrm{S}\beta^{-1} + \gamma^{-1}\mathrm{S}\gamma^{-1},$$

where α, β and γ are rectangular vectors with lengths a, b and c respectively. Recall that rectangular vectors are perpendicular so they can be assumed to lie on the x, y and z axis respectively. In addition, if the length of α is a, then the length of α^{-1} is $1/a$ by equation (5). Similar relations hold for β and γ, so that

$$\alpha^{-1} = \frac{1}{a}\hat{x}, \quad \beta^{-1} = \frac{1}{b}\hat{y}, \quad \text{and} \quad \gamma^{-1} = \frac{1}{c}\hat{z}.$$

This gives

$$\phi\rho = \phi = \alpha^{-1}\mathrm{S}\alpha^{-1}\rho + \beta^{-1}\mathrm{S}\beta^{-1}\rho + \gamma^{-1}\mathrm{S}\gamma^{-1}\rho.$$

Considering only the α^{-1} case,

$$\mathrm{S}\alpha^{-1}\rho = \alpha^{-1}\cdot\rho = \frac{x}{a} \quad \Rightarrow \quad \alpha^{-1}\mathrm{S}\alpha^{-1}\rho = \frac{x}{a^2}\hat{x}$$

The other terms are similar so that

$$\phi\rho = \left(\frac{x}{a^2}\hat{x} + \frac{y}{b^2}\hat{y} + \frac{z}{c^2}\hat{z}\right)$$

Then equation (12) gives

$$\mathrm{S}\rho\phi\rho = \rho\cdot\phi\rho = (x\hat{x} + y\hat{y} + z\hat{z})\cdot\left(\frac{x}{a^2}\hat{x} + \frac{y}{b^2}\hat{y} + \frac{z}{c^2}\hat{z}\right)$$

$$= \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

which is the equation of an ellipsoid, as claimed.

The goal of Hamilton's above mentioned paper was to derive the equation for the wave surface of a biaxal crystal (equation(1)) with quaternions. His first step was to seek out the direction of wave propagation. In most materials, light propagates in a direction perpendicular to the wave front, but this is not the case within biaxal crystals due to their multiple refractive indices. To find the direction of propagation, Hamilton used the displacement of the crystal molecules thought to be caused by the wave. He also used the slowness of the wave's propagation, which is perpendicular to the wave front (Figure 6). Furthermore, since a crystal can be considered an elastic medium, the displacement meets with a restoring force, given by some elastic constant times the displacement (Hooke's law). However, this restoring force is not necessarily in the opposite direction as the displacement. This may seem to violate Newton's third law ("If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction"; Thornton and Marion, 49), but this law only applies to central forces (forces between objects which act on a line connecting the objects). Velocity dependent forces (such as the one considered here which must propagate at the velocity of light) do not obey the third law. In a biaxal crystal, the elastic constant is different depending on the direction of displacement and is indicative of of the direction of propagation. By denoting the part of the displacement which is parallel to the wave surface $\delta\rho$, the elastic force has the form $\phi^{-1}\delta\rho$. To see this, let ϕ be as defined above and $\delta\rho = \delta x\hat{x} + \delta y\hat{y} + \delta z\hat{z}$. Then,

$$\phi^{-1}\phi\delta\rho = \delta\rho \quad \text{and} \quad \phi\delta\rho = \left(\frac{\delta x}{a^2}\hat{x} + \frac{\delta y}{b^2}\hat{y} + \frac{\delta z}{c^2}\hat{z}\right)$$

$$\Rightarrow \phi^{-1} = \alpha\mathrm{S}\alpha + \beta\mathrm{S}\beta + \gamma\mathrm{S}\gamma \Rightarrow \phi^{-1}\delta\rho = a^2\delta x\hat{x} + b^2\delta y\hat{y} + c^2\delta z\hat{z} = k\delta\rho,$$

if k is considered to be the elastic constant which varies in each direction. Hamilton did not use vector notation, but to avoid confusion from this point on vectors will be specified with the $(\vec{\ })$ symbol. From Figure 6 it can be seen that the tangential (to the wave surface) component of the elastic force has the direction of $\vec{\delta\rho}$ an the normal component is in the $\vec{\mu}^{-1}$ direction. Furthermore, if $\vec{\delta\rho}$ is considered a unit vector, the tangential component can be written $\mu^{-2}\vec{\delta\rho}$ (note that μ^{-2} is a scalar by (6)). Given this,

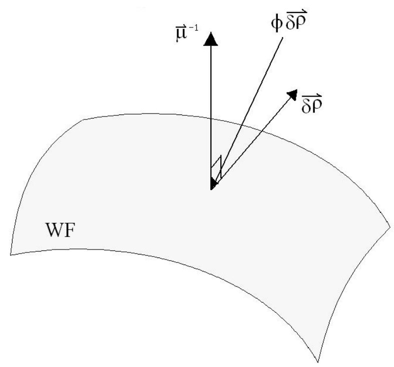

Figure 6: WF represents a wave front moving in the $\vec{\mu}^{-1}$ direction.

and the fact that the vector representing the elastic force is equal to the sum of its normal and tangential components, it can be concluded that the normal part ($\vec{F}_{normal}$) is

$$\vec{F}_{normal} = \vec{F}_{elastic} - \vec{F}_{tangential} = \phi^{-1}\vec{\delta\rho} - \mu^{-2}\vec{\delta\rho} = (\phi^{-1} - \mu^{-2})\vec{\delta\rho}.$$

But as previously observed this vector is in the direction of $\vec{\mu}^{-1}$, so it can also be denoted $\delta m\vec{\mu}^{-1}$ where δm is a scalar. Equating the two equations for the normal component of the elastic force and solving for $\vec{\delta\rho}$ gives

$$\delta m\vec{\mu}^{-1} = (\phi^{-1} - \mu^{-2})\vec{\delta\rho} \quad \Rightarrow \quad \vec{\delta\rho} = (\phi^{-1} - \mu^{-2})\delta m\vec{\mu}^{-1}.$$

Now by recalling that $\vec{\mu}^{-1}$ is perpendicular to $\vec{\delta\rho}$ and employing the quaternion property given in (7), $\mathrm{S}\vec{\mu}^{-1}\vec{\delta\rho} = 0$, but $\mathrm{V}\vec{\mu}^{-1}\vec{\delta\rho} = 0$ has the direction of wave propagation which has been sought. So, eliminating δm by writing

$$\vec{\tau}\delta m = \vec{\mu}^{-1}\vec{\delta\rho} \quad \Rightarrow \quad \vec{\tau} = \vec{\mu}^{-1}(\phi^{-1} - \mu^{-2})\vec{\mu}^{-1},$$

gives a quaternion $\vec{\tau}$, with no scalar part (hence the vector symbol) in the direction of true displacement within the crystal. The fact that the scalar part of $\vec{\tau}$ is zero gives rise to

$$\mathrm{S}\vec{\mu}^{-1}(\phi^{-1} - \mu^{-2})\vec{\mu}^{-1} = 0 \tag{10}$$

which can easily be identified as a quadric surface (8) since

$$\mathrm{S}\vec{\mu}^{-1}(\phi^{-1} - \mu^{-2})\vec{\mu}^{-1} = \mathrm{S}\vec{\mu}^{-1}\phi^{-1}\vec{\mu}^{-1} - \mathrm{S}\vec{\mu}^{-1}\mu^{-2}\vec{\mu}^{-1} = 0$$

$$\Rightarrow \quad \mathrm{S}\vec{\mu}^{-1}\phi^{-1}\vec{\mu}^{-1} = [(\mathrm{T}\vec{\mu}^{-1})^2]^2 \ \Rightarrow \ \mathrm{S}\vec{\mu}^{-1}\phi^{-1}\vec{\mu}^{-1} = \text{constant}.$$

Hamilton identified (10) as the equation of an index surface, since the direction of true displacement is related to the refractive index. Next, letting the vector perpendicular to the wave surface be $\vec{\rho}$, Hamilton used physical principles to relate $\vec{\mu}^{-1}$ to $\vec{\rho}^{-1}$, and the methods of quaternion transformations to transform (10) into a function of $\vec{\rho}$ and conclude that the result was the quaternion equation for the wave surface. This equation is

$$\mathrm{S}\vec{\rho}^{-1}(\phi - \rho^{-2})\vec{\rho}^{-1} = 0 \tag{11}$$

By comparing (11) to (1), the advantages in quaternion notation are evident. Also, the ease with which Hamilton was able to arrive at his equation shows the strength of quaternions. In the preceding description, the explanation of the physical properties made up most of the work. Once these properties had been described by quaternion equations, Hamilton essentially only needed three steps. First, he turned the expressions for quantities known to be perpendicular to the wave into a quaternion equation which could be solved to give the direction of displacement within the crystal. Next, he used this equation to write (10), the equation of a surface representing the refractive index. Finally, he transformed (10) into the Fresnel's wave surface in quaternion form.

It may be that by revisiting previous mathematical findings after his discovery of quaternions convinced Hamilton that those discoveries would have been made sooner or more easily had quaternions always been used. It is also likely that the simplicity and utility of quaternion mathematics made it impossible for Hamilton to suspect that they were just one of a number of equally effective systems rather than a fundamental method for describing nature. Quaternions allowed Hamilton to solve problems which he could not solve by any other method at the time, so his devotion to them hardly seems unjustified. Even though the importance of quaternions can be debated, the significance of Hamilton's contribution's to mathematical optics and to science in general have guaranteed that the fame he had in life will continue well into the foreseeable future.

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*These documents are available for download from Trinity College of Dublin's School of Mathematics website: http://www.maths.tcd.ie/pub/HistMath/people/Hamilton/Optics.html