Pathological Functions in the 18th and 19th Centuries

Mame Maloney
The University of Chicago
maloney@uchicago.edu
303 Edgewood Avenue
Clemson, SC 29631

The ghost of a crisis emerges when both Körner, in his *Fourier Analysis*, and Lakatos, in *Proofs and Refutations*, recall an anecdote about "frightened and horrified" reactions to nowhere differentiable functions (Lakatos (1976), 19). Körner narrates:

> Many mathematicians held up their hands in (more or less) genuine horror at 'this dreadful plague of continuous nowhere differentiable funtions' (Hermite), with which nothing mathematical could be done. They complained that the hypotheses required to avoid what they called 'pathological functions' spoilt the elegance of classical analysis and that concentration on such functions would spoil the geometric intuition which is at the heart of analysis. (Körner (1989), 42)

As Körner tells the story, mathematicians like Charles Hermite and Henri Poincaré rejected nowhere differentiable functions on the grounds that analysis was in the business of talking about differentiable functions. Both Körner and Lakatos characterize these nay-sayers as striving to protect the simplicity and elegance of analysis. We get the sense that mathematicians like Hermite and Poincaré saw absolutely no value in nowhere differentiable functions, which came at the cost of the beauty and universal applicability of their theorems. The historical accuracy of these authors' claims is far from certain, though; the story of Hermite's "fright and horror" is common enough, but no reference is ever given. This opens the question of whether Hermite's and Poincaré's comments were just parlor talk, or if they ever championed these positions in their work. The history of function theory is fraught with such mysteries.

Dirichlet's characteristic function of the rationals is a particularly abstruce pathological function. This everywhere-discontinuous function is defined as

$$D(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & \text{else} \end{cases}$$

Dirichlet's own opinions about his function are ambiguous, and Kleiner, Luzin, Youschkevitch, and Lakatos each give differing narratives.

Kleiner credits Dirichlet with providing a "clear understanding of the function concept," which Kleiner sees as necessary in order to prove a theorem giving sufficient conditions for Fourier-representability. Kleiner and Luzin both quote the following passage as Dirichlet's definition of function, but, unfortunately, neither author refers to an original article by Dirichlet.

> y is a function of a variable x, defined on an interval $a < x < b$, if to every value of the varible x in this interval there corresponds a definite value of the variable y. Also, it is irrelevent in what way this correspondence is established. (Luzin (1998), 264)

This definition compares quite favorably with the modern definition. Today, we define a function as a correspondance f between two sets A and B, such that each $a \in A$ is assigned a unique $f(a) \in B$. Neither the modern definition nor Dirichlet's definition relies on the geometric or analytic representation in any way. The leap from geometric or analytic representability to arbitrary correspondance is exactly what made modern point-set considerations possible.[1] Dirichlet's leap from the typical 19th-century preoccupation with analytic representability to the modern conception is, naturally, forward-thinking. Luzin says, "This definition immediately clarified a great many hitherto at best vaguely understood phenomena of mathematical analysis," (Luzin (1998), 264). Many authors and mathematicians join Luzin in crediting Dirichlet with the current, general definition of function.

Youschkevitch contradicts Luzin's depiction of Dirichlet as forward-thinking genius. He cites the following, from Dirichlet's 1837 *Über die Darstellung ganz willkürlicher Funktionen durch Sinus- und Cosinusreihen* (On the Presentation of all arbitrary Functions by means of Sine- and Cosine-series), as Dirichlet's definition of function:

> One means by a and b two fixed values and by x a variable magnitude, which bit by bit takes on all values lying between a and b. To each x corresponds a unique y so that as x runs continuously between a and b, $y = f(x)$ likewise gradually changes, so that y is called a continuous function of x on this interval. It is certainly not necessary that y depend on x according to the same rule

[1] Manheim claims that if mathematicians had stuck with Euler's conception of function, i.e. that a geometrically representable function must be analytically representable, mathematicians would have not have been able to make the necessary abstractions which allow for topology as we know it today (Manheim (1964)).

> throughout this whole interval; indeed, one does not even need a relationship expressible through mathematical operations. Presented geometrically, thinking of x as the abscissa (x-axis) and y as the ordinate (y-axis), a continuous function looks like a connected curve on which a single point corresponds to each x between a and b. This definition dictates no law to the individual parts of the curve; one can imagine this curve as being composed of various pieces or even as randomly drawn. It follows from this that such a function can be considered as totally specified for an interval only when the components of the function are made to follow the same laws as one another. So long as one has specified a function on only a part of an interval, its determination on the rest of the interval remains totally arbitrary. (Youschkevitch (1975), 78)[2]

Youschkevitch impresses upon the point that Dirichlet only defines *continuous* functions here. This is the only conceptual difference between the definition given here and that given by Luzin.

Youschkevitch asks, "Why did both these scholars [Lobatchevsky and Dirichlet] think it expedient to restrict their definitions with continuous functions?" (Ibid., 79). He provides a simple rationale, borrowed from Medvedev: continuous functions were important, so they captured attention and focus. Youschkevitch implies that Dirichlet did not have a fundamental problem with discontinuity, but instead simply did not bother mentioning discontinuous functions explicitly in his definitions. Youschkevitch backs up this hypothesis by showing that Dirichlet was comfortable working with discontinuous functions in his work. Both Dirichlet and Lobatchevsky authored theorems showing that functions with isolated discontinuities were representable by Fourier series (ibid., 78). Indeed, in Dirichlet's 1829 *Sur la*

[2]Original quotation: "Man denke sich unter a und b zwei feste Werthe und unter x eine veränderliche Grösse, welche nach und nach alle zwischen a und b liegenden Werthe annehmen soll. Entspricht nun jedem x ein einziges, endliches y, und zwar so, dass, während x das Intervall von a bis b stetig durchläuft, $y = f(x)$ sich ebenfalls allmächlich verändert, so heisst y eine stetige oder continuirliche Function von x für dieses Intervall. Es ist dabei gar nicht nöthig, dass y in diesem ganzen Intervalle nach demselben Gesetze von x abhángig sei, ja man braucht nicht einmal an eine durch mathematische Operationen ausdrückbare Abhängigkeit zu denken. Geometrisch darstellt, d.h. x und y als Abszisse und Ordinate gedacht, erscheint eine stetige Function als eine zusammenhängende Curve von der jeder zwischen a und $b$enthaltenen Abszisse nur ein Punkt entspricht. Diese Definition schreibt den einzelnen Theilen der Curve kein gemeinsames Gesetz vor; man kann sich dieselbe aus den verschidenartigsten Theilen zusammengesetzt oder ganz gesetzlos gezeichnet denken. Es geht hieraus hervor, dass eine solche Function für ein Intervall als vollständig bestimmt nur dann anzusehn ist, wenn sie entweder für die einzelnen Theile desselben geltenden Gesetzen unterworfen wird. So lange man über eine Function nur für einen Theil des Intervalls bestimmt hat, bleibt die Art ihrer Forsetzung für das übrige Intervall ganz Willkür überlassen."
The translation given above is credited to my friend Maya Vinokour.

convergence..., he says:

> The preceeding considerations prove in a rigorous manner that, if the function $\varphi(x)$, all of whose values are finite and determinite, only has a finite number of discontinuities between the limits $-\pi$ and π, and also if it only has a dtermi-nite number of maima and minima between these same limits, the series [cross-referenced to elsewhere in the text]
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> $$\frac{1}{2\pi}\int \varphi(\alpha)\partial\alpha + \frac{1}{\pi}\left\{ \begin{array}{l} \cos x \int \varphi(\alpha \cos \alpha\partial\alpha + \cos 2x \int \varphi(\alpha) \cos 2\alpha\partial\alpha ... \\ \sin x \int \varphi(\alpha) \sin \alpha\partial\alpha + \sin 2x \int \varphi(\alpha) \sin 2\alpha\partial\alpha ... \end{array} \right\}$$
>
> whose coefficients are definite integrals dependent on the function $\varphi(x)$ is convergent and has a general value expressed by $\frac{1}{2}[\varphi(x+\varepsilon) + \varphi(x-\varepsilon)]$, where ε is infinitely small. (Dirichlet (1829), 168–169) [3]

In other words, the Fourier series of a function converges when the function is finite and has a finite number of maxima, minima, and discontinuities. Clearly, Dirichlet is comfortable talking about discontinuous functions. For Youschkevitch, Dirichlet's treatment of discontinuous functions adds a layer of ambiguity to Dirichlet's definition of function, rather than indicating that this definition did not accurately represent Dirichlet's views.

Lakatos goes even farther than Youschkevitch from Luzin's account of Dirichlet as free mathematician. In the context of *Proofs and Refutations*, Dirichlet is a monster-barrer who invented, and then barred, his own monster. Lakatos says,

> The trouble again is that Dirichlet still held that all genuine functions are in fact Fourier-expandable—he devised this 'function' explicitly as a monster. According to Dirichlet his 'function' is an example not of an 'ordinary' real function, but of a function which does not really deserve the name. (Lakatos (1976), 151)

Lakatos does not present Dirichlet as denying that the function $D(x)$ can be reasoned about mathematically; instead, Lakatos portrays Dirichlet as merely saying that $D(x)$ is not a function in the proper sense.

[3] "Les considérations précédentes prouvent d'une manière rigoreuse que, si la fonction $\varphi(x)$, dont toutes les valeurs sont supposées finies et déterminées, ne présente qu'un nombre fini de solutions de continuité entre les limites $-\pi$ et π, et si en outre elle n'a qu'un nombre déterminé de maxima et de minima entre dex mêmes limites, la série [cross-referenced to elsewhere in the text] dont les coëfficiens sont des intégrales défines dépendantes de la fonction $\varphi(x)$ est convergente et a une valeur généralement exprimé par $\frac{1}{2}[\varphi(x+\varepsilon) + \varphi(x-\varepsilon)]$, où ε désigne un nombre infiniment petit."

NB: "solutions de continuité" means points of discontinuity.

Furthermore, Lakatos claims that not only did Dirichlet not believe in a more general definition of function, he was conceptually distant from such a definition. As evidence that Dirichlet did not comprehend the function-concept Luzin attributes to him, Lakatos shows that that Dirichlet, in his *Über die Darstellung...*, describes piecewise discontinuous functions as assuming two different values at the points of discontinuity (ibid., 151).

Of the four accounts given above, Lakatos' is most accurate, but he fails to pick up on the discrepancy Youschkevitch notes between Dirichlet's definitions and the structures Dirichlet chooses to work with.

Dirichlet's discussions of functions are primarily contained in his 1829 *Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitrarie entre des limites données* (On the convergence of trigonometric series which serve to represent an arbitrary function between given limits), his 1837 *Sur les séries dont le terme général dépend de deux angles, et qui servent à exprimer des fonctions arbitraries entre des limites données* (Over series of which the general term depends on two angles, and which serves to formulate arbitrary functions between given limits), and his 1837 *Über die Darstellung ganz willkürlicher Funktionen durch Sinus- und Cosinusreihen* (On the Presentation of all arbitrary Functions by means of Sine- and Cosine-series).

The word "arbitrary" in all three of these titles is not as strong as a definition, but it does suggest that Dirichlet believes that functionhood is meaningfully tied up with representability by a Fourier series. Dirichlet even opens his 1829 article by saying, "Series of sines and cosines, by means of which one can represent an *arbitrary function* in a given interval, enjoy among other remarkable properties that of being convergent," (Dirichlet (1829), 157; emphasis added).[4] Throughout this paper, he refers to functions developed by series of sines and cosines as "arbitrary." When he applies restrictions to these "arbitrary" functions, he specifies that they must have determinite, finite values in an interval, and achieve only a finite number of maxima, minima, and points of discontinuity. In the 1837 *Sur les Séries...*,

[4] "Les séries de sinus et de cosinus, au moyen desquelles on peut représenter une fonction arbitrarie dans un intervalle donné, jouissent entre autres propriétés remarquables aussi de celle d'être convergentes."

he is a little more precise about what he means by "arbitrary," saying, "arbitrary functions, i.e., functions which are not subject to any analytic law..." (Dirichlet (1837), 288)[5] But then again, he is just as loose with this term later in the paper, when he defines a continuous function and claims that it is "arbitrary," just like in the Youschkevitch article. Dirichlet says,

> The series which we will consider in this memoir are ordinarily followed by functions of a particular form, functions which Legendre first made use in his beautiful research on the attraction of ellipsoids of revolution and the face of the planets. These functions enjoy a great number of remarkable properties, and the series from which they are made are fit to represent arbitrary functions between certain limits. The generality of this last proposition not being sufficiently settled by the considerations which bring about developments of this type in this theory of the attraction of spheriods, one has sought to prove this last claim in a direct manner independent of this theory.
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> If one designates by P_n the coefficient of α^n in the value developed of the radical
>
> $$\frac{1}{\sqrt{1-2\alpha(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi'-\varphi))+\alpha^2}}$$
>
> the proposition that it acts, as formulated by the equation:
>
> $$f(\theta,\varphi)=\frac{1}{4\pi}\sum_{n=0}^{n=\infty}(2n+1)\int_0^{\pi}d\theta'\sin\theta'\int_0^{2\pi}P_n f(\theta',\varphi')d\varphi'$$
>
> which has place for all the values of θ and φ comprises between the limits $\theta=0$ and $\theta=\pi$, $\varphi=0$ and $\varphi=2\pi$, the function $f(\theta,\varphi)$ stays entirely arbitrary between these limits and is only subjected to not becoming infinite. (Ibid., 283–284)[6]

[5] "...fonctions arbitrarires, c'est-à-dire des fonctions qui ne sont assujetties à aucune loi analytique."

[6] Original passage: "Les séries que nous nous proposons de considérer, dans ce Mémoire, sont ordonées suivant des fonctions d'une forme particulière, fonctions dont Legendre a le premier fait usage dans ses belles recherches sur l'attraction des ellipsoïdes de révolution et sur la figure des planètes. Ces fonctions jouissent d'un grand nombre des propriétés remarquables et les séries qui en sont formées, sont propres à représenter des fonctions arbitraries entre certaines limites. La généralité de cette dernière proposition n'ayant pas été jugée suffisamment établie théorie de l'attraction des sphéroïdes, on a cherché à la prouver d'une manière directe et indépendante de cette théorie.

"Si l'on désigne par P_n le cofficient de α^n dans la valeur développée du radical: $\frac{1}{\sqrt{1-2\alpha(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi'-\varphi))+\alpha^2}}$ la proposition dont il s'agit, sera exprimée par l'équation: $f(\theta,\varphi)=\frac{1}{4\pi}\sum_{n=0}^{n=\infty}(2n+1)\int_0^{\pi}d\theta'\sin\theta'\int_0^{2\pi}P_n f(\theta',\varphi')d\varphi'$ qui a lieu pour toutes les valeurs de θ et de φ comprises entre les limites $\theta=0$ et $\theta=\pi$, $\varphi=0$ et $\varphi=2\pi$, la fonction $f(\theta,\varphi)$ restant entièrement arbitrarie entre ces limites et étant seulement assujettie à ne pas devenir infinie."

In this passage, he claims to be able to express any functions with this series of elliptic functions. But $f(\theta, \varphi)$ will always be continuous. Still, Youschkevitch is wrong to say that Dirichlet stuck to the idea that all functions are continuous, since, as we see above, Dirichlet uses the word "arbitrary" just about every chance he gets.

Does Dirichlet make any meaningful restrictions to his function concept? Lakatos claims that Dirichlet's construction of the characteristic function of $\mathbb{Q}$ is just such an example of a meaningful, serious restriction to Dirichlet's function concept. Dirichlet's famous construction appears in his 1829 paper, and goes as follows:

> It is necessary that the function $\varphi(x)$ is such that, if a and b are any two quantities between $-\pi$ and π, one can always find between a and b two other close-together quantities r and s such that $\varphi(x)$ is continuous on the interval from r to s. This restriction is obviously necessary, considering that the different terms of the series are definite integrals. The integral of a function doesn't signify anything besides that the function satisfies the above condition. A function that does not meet this condition is produced by taking $\varphi(x)$ equal to a determinate constant c when the variable x is rational, and equal to another constant d when this variable is irrational. The function thus defined has two finite and determinite values for every value of x, but nevertheless one does not know how to represent it by a series, since hte different interrals which enter into the series lose all significance in this case. The only restrictions to which $\varphi(x)$ is subjected are that specified above and that of not becoming infinite; the preceeding discussion applies to all other cases.(Dirichlet (1829), 169)[7]

Lebesgue, as late as 1905, begrudgingly adimts that after Dirichlet and Riemann, the definition of function no longer concerns the exact procedure which establishes a correspondence between variables. But still, to him, "true" functions are analytically representable.

[7] "Il est nécessaire qu'alors la fonction $\varphi(x)$ soit telle que, si l'on désigne par a et b deux quantités quelconques comprises entre $-\pi$ et π, on puisse toujours placer entre a et b d'autres quantités r et s assez rapprochées pour que la fonction reste continue dans l'intervalle de r à s. On sentira facilement la nécessité de cette restriction en considérent que les différens termes de la série sont des inégrales défines et remontant à la notion fondamentale des intégrales. On verra alors que l'intégrale d'une fonction ne signifie quelque chose qu'autant que la fonction satisfait à la condition précédemment énoncée. On aurait un exemple d'une fonction qui ne remplit pas cette condition, si l'on supposait $\varphi(x)$ égale à une constante déterminée c lorsque la variable x obtient une valeur rationelle, et égale à une autre constante d, lorsque cette variable st irrationnelle. La fonction ainsi définie a des valeurs finies et déterminées pour toute valuer de x, et cependant on ne saurait la substituer dans la série, attendu que les différentes intégrales qui entrent dans cette série, perdroient toute signification dans ce cas. La restriction que je viens de préciser, et celle de ne pas devenir infinie, sont les seules auxquelle la fonction $\varphi(x)$ soit sujette et tous les cas qu'elles n'excluent pas peuvent être ramenés à ceux que nous avons considérés dans ce qui précède."

He argues that this restriction is neither needlessly conservative nor arbitrary, becasue only analytically representable functions are used effectively in math. Even in theories such as Riemannien integration, in which it is not necessary to ensure analytic representability, the functions mathematicians work with are, in fact, always analytically representable (Lebesgue (1905), 139).[8]

Lebesgue's commentary cuts to the heart of the attitude against everywhere-discontinuous and otherwise pathologial functions expressed by mathematicians like Hermite, Poincaré, and Dirichlet. Simply put, pathological functions were not useful to these mathematicians, so they were not worth assigning the same status as analytically representable functions. Furthermore, the precise definition of function factors very little into these mathematicians' works. Lebesgue sees the more precise and general definition of function, which we essentially use today, as a frivolity at best and a liability at worst. Mathematicians against the pathological functions did not discuss them in depth because these functions did not factor meaningfully into their work.

[8] "Bien que, depuis Dirichlet et Riemann, on s'accorde généralement à dire qu'il y a fonction quand il ya correspondance entre un nombre y et des nobmres $x_1, x_2, \ldots, x_n$, sans se préoccuper du procédé qui sert à établir cette correspondance, beaucoup de mathématiciens semblent ne considérer comme de vraies fonctions que celles qui sont établies par des correspondances analytiques. On peut penser qu'on introduit peut-être ainsi une restriction assez arbitraire; cependant il est certain que cela ne restreint pas pratiquement le champ des applications, parce que, seules, les fonctions représentables anlytiquement sont effectivement employées jusqu'à present.

"Dans certaines théories générales, dans la theéorie de l'intégration au sens de Riemann, par exemple, on ne se préoccupe pas de savoir si les fonctions que l'on considère sont ou non représentables analytiquement. Mais cela ne veut pas dire qu'elles ne le sont pas toutes et, dans tous les cas, quand on applique effectivement ces théories, c'est toujours sur des fonctions représentbles analytiquement qu'on opère."

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