The *Suan shu shu* and the *Nine Chapters on the Mathematical Art*: A Comparison

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1 Introduction

The *Jiu zhang suan shu* (translated as the *Nine Chapters on the Mathematical Art*) has been called "the most important text of ancient Chinese mathematics" (Dauben, "Suan shu shu" 94). Believed to have been compiled during the Former Han dynasty – most likely during the first century BC – as a resource for men studying for the Chinese civil service examinations, the impact of this text on the Asian mathematics community is widespread in that "countless works have been inspired by the classification of mathematics in nine chapters or have borrowed their vocabulary and their resolutory methods from the [*Jiu zhang suan shu*]" (Martzloff 128).

According to the Chinese mathematician Liu Hui, the *Nine Chapters* was compiled by Zhang Cang – a lofty government official who lived "c. 250-152 B.C." – and Geng Shouchang – another high-ranking government official who "[flourished] c. 57-52 B.C." (Cullen). Liu, who wrote commentary on the *Nine Chapters* in 263 AD, explains in his introduction to this text that one of the emperors of Qin had a book called the *Zhou li* (*Rites of Zhou*) destroyed. The *Zhou li* was a Confucian book with a section dedicated to mathematics called the *Jiushu* (*Nine Arithmetical Arts*). After the *Jiushu* was destroyed, Zhang and Geng reconstructed it, "[w]orking on the basis of the remains of incomplete old manuscripts" (Kangshen et al. 53). In addition to recreating the information from the *Nine Arithmetical Arts*, Zhang and Geng also "revised and supplemented [the information from the text]" (Kangshen et al. 53). This reconstruction of the *Jiushu* became known as the *Nine Chapters on the Mathematical Art*.

While the *Nine Chapters* was thought to have only been based on the *Nine Arithmetical Arts*, archaeological discoveries from 1983 to 1984 have altered this assumption. During this period, a series of bamboo strips were found in three tombs being excavated at Zhangjiashan in China's Hubei province. These bamboo strips together formed a mathematical work that came to be called the *Suan shu shu* (*A Book on Arithmetic*). The *Suan shu shu* is believed to have been written in the early second century BC as a "ready reference for government bureaucrats of the Qin and early Han dynasties" (Dauben, "Suan shu shu" 92), and has no known authors.

Since the discovery of the *Suan shu shu*, there has been debate among the scholarly community about how much influence the *Suan shu shu* may have exerted on the *Nine Chapters*. No one argument on this matter has been particularly endorsed. This paper seeks to determine the influence of the *Suan shu shu* on the *Nine Chapters of the Mathematical Art* by examining the textual similarities and differences between the two texts, as well as by considering the historical circumstances surrounding the texts. From this research, it can be concluded that the *Nine Chapters* is at least partially based on the *Suan shu shu*; the *Nine Chapters* was not only compiled using manuscripts of the *Nine Arithmetical Arts*, but also slips from the *Suan shu shu*. The *Nine Chapters* took much of the material in the *Suan shu shu* and provided a larger repertoire of examples that were more advanced than those of the *Suan shu shu*. Some of the problems in the *Nine Chapters* were adapted from problems in the *Suan shu shu* in order to make these problems more concise and comprehensible. As explained by Liu Hui, those who compiled the *Nine Chapters* "supplemented [the text]" from the *Jiushu*; likewise, those who compiled the *Nine Chapters* supplemented the content they extracted from the *Suan shu shu*, adding mathematical discoveries that were not known at the time of the *Suan shu shu*, or were not included in the *Suan shu shu* for other reasons. The wider scope of the *Nine Chapters* is evident from the number of problems that each text has: the *Nine Chapters* contains 246 problems, while the *Suan shu shu* contains 68.

The conclusions in this paper may not have reached the highest level of accuracy, for English translations of these texts were utilized in this research instead of texts written in the original Chinese. The fact that these translations were done by different scholars could also affect the accuracy of this paper's conclusions. In his translation of the *Suan shu shu*, Joseph W. Dauben writes fractions with slashes, but this is corrected here according to modern scholarly notation. Dauben inserts missing text into the problems of the *Suan shu shu* using parentheses, which have been changed to brackets in this paper.

Another feature of these texts that should be noted is that they both contain instructions to "lay down" (such as in problem 1.6 of the *Nine Chapters* (Kangshen et al. 64), with "1" signifying the chapter of the *Nine Chapters* in which this problem is found, and "6" signifying the number of the problem in that chapter) or "put down" (such as in problem 15 of the *Suan shu shu* (Dauben, "Suan shu shu" 119)). This is in reference to counting rods, which were placed on counting boards in order to perform mathematical operations. These counting boards were used "from around 500 BC to 1500 AD" in China until the widespread adoption of the abacus (Kangshen et al. 11).

2 Analysis

2.1 Chapter 1 of the *Nine Chapters on the Mathematical Art*: Field Measurement (*Fangtian*)

The name of Chapter 1 of the *Nine Chapters* matches the title of problem 53 of the *Suan shu shu*: *Fangtian*. Problem 53 of the *Suan shu shu* seeks to find the dimensions of a square field that yield an area of 1 *mu*. The most likely reason that Chapter 1 of the *Nine Chapters* is called *Fangtian* is that many of the problems in the chapter are dedicated to finding the areas of fields. It is possible that the authors of the *Nine Chapters* were using the *Suan shu shu* as a guide for their own work, and took the title of problem 53 from the *Suan shu shu* as the name of their first chapter since problem 53 of the *Suan shu shu* is one of the few problems in the *Suan shu shu* concerned with fields and the area of fields.

In the *Suan shu shu* and the *Nine Chapters*, problems follow the same format: first the exposition for the problem is given, next the question which the problem is trying to answer is asked, then the answer to the problem is given, and finally the explanation is written of how the answer was reached. As there are no known Chinese mathematical classics written before the *Nine Chapters* other than the *Suan shu shu*, it can be hypothesized that the *Nine Chapters* adopted the format of these problems from the *Suan shu shu*.

Although the format of the problems across both texts are essentially the same, differences can be seen in their explanations of the problems' methods. While there are few general rules given in the *Suan shu shu* as methods, a general rule is given for every problem in the *Nine Chapters*. The rules in the *Nine Chapters*, unlike those of the *Suan shu shu*, are named as well. Thus, the methods as presented in the *Nine Chapters* are more organized than those presented in the *Suan shu shu*. The order of the problems in the *Nine Chapters* is also more organized than that of the *Suan shu shu*. While the problems of the *Suan shu shu* are somewhat ordered by topic (e.g., problems 1-8 and 10 involve operations with fractions), there is no grouping of the problems into sections or chapters as in the *Nine Chapters*. In the *Nine Chapters*, the problems are not only organized into different chapters based on topic, but are grouped together in each chapter by their methods or rules. Problems 3 and 4 of Chapter 1 are grouped together because the method for both is the Rule for Rectangular Fields Measured by *li*. Problems 1.3 and 1.4, as well as the method for the problems, are written in the *Nine Chapters* as follows:

> **[Problem 3]**
>
> Now given a field, 1 *li* in breadth and 1 *li* in length. Tell: how much field?
>
> Answer: 3 *qing* 75 *mu*.
>
> **[Problem 4]**
>
> Given another field, 2 *li* in breadth and 3 *li* in length. Tell: how much field?

Answer: 22 *qing* 50 *mu*.

The Rule for Rectangular Fields Measured by *li*

Multiply the number of *li* in breadth by that in length to obtain the product. Multiplying by 375 gives the number of *mu*. (Kangshen et al. 63)

Problem 1.3 bears a remarkable similarity to problem 68 of the *Suan shu shu*:

***Li tian*: Farmland [Measured in] Li**

The *Li Tian* Method [The Method of Measuring Farmland in *Li*] says: *li* times *li* are [square] *li*. If the width and length are both 1 *li*, then put [down on the counting board] 1 standing for 3, also multiply by 5 three times; the result is a field of 3 *qing* 75 *mu*. If its width and length are not equal, first multiply the [amounts of] *li* together, then multiply [the result] by 3, again multiply by 5, three times, which completes it [the *li tian* method].

[68a] Now there is a width of 220 *li*, and a length of 350 *li*, which constitute a field of 288,750 *qing*. To officially distribute land, use this method. [Another method] says: *li* multiplied by *li* result in [square] *li*. First multiply by 3 then multiply by 5 three times which is the number of *qing* and *mu*. Yet another [method] says: *li* times *li* is *li*. Using [*li*], next multiply by 25, then by 3, which also [gives] its amount [area] in *qing* and *mu*. That is to say, if the width is 1 *li*, and the length is 1 *li*, the field is 3 *qing* 75 *mu*. (Dauben, "Suan shu shu" 166-167)

The width and length, as well as the area, of the fields in problem 1.3 of the *Nine Chapters* and problem 68 of the *Suan shu shu* are identical. While the dimensions of these two problems are the same, problem 1.3 of the *Nine Chapters* is more condensed than problem 68 of the *Suan shu shu*. Problem 68 of the *Suan shu shu*, besides stating the dimensions of two separate fields in the same problem, explains the same method in four different ways. One method says to first multiply the width of the field by the length, and then to "multiply by 3 then multiply by 5 three times which is the number of *qing* and *mu*." Another method in problem 68, after multiplying the width in *li* by the length in *li*, advises to "multiply by 25, then by 3, which also [gives] its amount [area] in *qing* and *mu*." These methods are essentially equal, for in both the mathematician is multiplying the width by the length, and then multiplying that product by 375. The *Nine Chapters* condenses all of these methods into one in its Rule of Rectangular Fields Measured by *li*: "Multiply the number of *li* in breadth by that in length to obtain the product. Multiplying by 375 gives the number of *mu*." The method as written in the *Nine Chapters* is more organized than the method as written in the *Suan shu shu* not only because it avoids repetition, but also because it states in clear terms that the mathematician is to multiply the width (or "breadth") by the length. The *Suan shu shu* indicates this step in somewhat unclear terms, saying instead that "*li* times *li* are [square] *li*," or similarly "*li* times *li* is *li*." This step is repeated several times in the *Suan shu shu* while the *Nine Chapters* explains it once. In these problems, the *Nine Chapters* shows itself to be more coherent and concise than the *Suan shu shu*, and thus superior. The fact that some of the topics in this chapter, such as the rules for finding the areas of bowl-like fields and annular fields, are not included in the *Suan shu shu* suggests that the *Nine Chapters* is also more advanced and extensive than the *Suan shu shu*.

2.2 Chapter 2 of the *Nine Chapters on the Mathematical Art*: Millet and Rice (*Sumi*)

Chapter 2 of the *Nine Chapters* concerns Rule of Three problems, particularly in situations of trade. The Rule of Three is also known as the *Jinyou* Rule, with *jin you* literally meaning "there is/there are." While the term *jin you* is used in the *Suan shu shu* for its literal meaning and is not always used in that text's rule of three problems, Dauben comments that "[b]y the time the *Nine Chapters* was compiled, this [*jin you*] had become a technical phrase used systematically to introduce a very specific group of problems, namely those that involve the 'rule of three'" (Dauben, "Suan shu shu" 113). In the *Nine Chapters*, however, this rule is not only used to introduce rule of three problems; it is "also used to introduce virtually every problem in the book" (Dauben, "Suan shu shu" 9). The Rule of Three is written in Chapter 2 of the *Nine Chapters*:

The Rule of Three (*Jinyou* Rule)

> Take the given number [*suoyoushu*] to multiply the sought rate [*suoqiulu*]. [The product] is the dividend. The given rate [*souyoulu*] is the divisor. Divide. (Kangshen et al. 143)

Crossley et al. explains the *Jinyou* Rule:

> Every problem has two sets of rates, the ratios of which are equal. We use a to denote the *souyoulu* (given rate) and c to denote the *suoqiulu* (sought rate), which are the corresponding exchange rates for grains A and C respectively in the table [of the exchange rates]. b, the *suoyoushu* (given number), is the amount of grain A given in the problem. x, the *suoqiushu* (sought number), is the required amount of grain C. According to the Rule of Three this means: $x = b \times c \div a$. (Kangshen et al. 143)

Let us look at an example of a *jinyou* problem from this chapter:

> **[Problem 1]**
>
> Now millet, 1 *dou*, is required as hulled millet. Tell: how much is obtained?
>
> Answer: As hulled millet: 6 *sheng*.

> Method: Taking millet as required as hulled millet, multiply by 3, divide by 5. (Kangshen et al. 143)

Here, the “given number” (as it is called in the “Rule of Three” above) is 1 (from 1 *dou*). The sought rate is that of hulled millet, for which the rate given in the *Nine Chapters* is 30. The given rate is that of millet, which is 50. In problem 1, the rates are reduced from 30 and 50 to 3 and 5, respectively. The given rate, 1, is multiplied by the reduced sought rate, 3, and then is divided by the reduced given rate, which is 5. This gives us an answer of 0.6 *dou*, which is converted to 6 *sheng*.

The first time a problem using the Rule of Three is found in the *Suan shu shu* is Problem 17:

> ***Jin jia*: The Price of Gold**
>
> The price of [1] *liang* of gold is 315 *qian*. If now there is [*jin you*] 1 *zhu* [= $\frac{1}{24}$ *liang*], how many *qian* is it worth? [The answer] says: it is worth 13 and $\frac{1}{8}$ *qian*. The method says: put down [on the counting board] the number of *zhu* in one *liang* as divisor. Using the number of *qian* as the dividend, dividing the dividend by divisor yields the amount of *qian*. There are 24 *zhu* per *liang*, 384 *zhu* per *jin*, 11,520 *zhu* per *jun*, and 46,080 *zhu* per *shi* [weight]. (Dauben, “Suan shu shu” 122)

Here, the given number is 1 *zhu*; the sought rate is the number of *qian* in 1 *liang*, which is 315; the given rate is the number of *zhu* in 1 *liang*, which is 24 *zhu*. Thus, the method is to multiply 1 by 315, and then divide by 24.

Dauben states that in the *Nine Chapters*, “[v]irtually all such problems follow the same format: ‘There are x of something at a price or value y. *Jin you* [now there are] z of that same something, what is the price?’” (Dauben, “Suan shu shu” 96) Almost all of the problems in the *Suan shu shu* that use the Rule of Three also follow this format, such as problem 17 (given above). It is possible that the *Nine Chapters* took this format from the *Suan shu shu*.

2.3 Chapter 3 of the *Nine Chapters on the Mathematical Art*: Proportional Distribution (*Cuifen*)

As its title suggests, Chapter 3 of the *Nine Chapters* focuses on proportional distribution. Proportional distribution, continued proportion, inverse proportion, direct proportion, and compound proportion are all included in this chapter. All of these problems involve the Rule of Three. Problem 11 of the *Suan shu shu* includes the term *cui fen* (meaning "proportional distribution"). Thus, the *Nine Chapters* may have intentionally taken one of the topics from the *Suan shu shu* – proportional distribution – and chosen to treat it more extensively, for there are only six problems in the *Suan shu shu* dedicated to proportional distribution while there are twenty problems involving proportional distribution in Chapter 3 of the *Nine Chapters*.

Dauben calls attention to the fact that problem 4 of this Chapter is very similar to problem 14 of the *Suan shu shu*. Problem 14 is another proportional distribution problem:

> ***Nu zhi*: Woman Weaving**
>
> There is a woman in the neighborhood who is displeased with herself [i.e. with her weaving], but happy that every day she doubles her weaving. In five days [she] weaves five *chi*. How much does she weave in the first day, and how much in every day thereafter? The answer: the first day she weaves $1\frac{38}{62}$ *cun*; then 3 and $\frac{14}{62}$ *cun*; then 6 and $\frac{28}{62}$ *cun*; then $12\frac{56}{62}$ *cun*; then 25 and $\frac{50}{62}$ *cun*. The method says: put down the values 2, 4, 8, 16, 32; add these together as the divisor; taking the 5 *chi*, multiply this by each of them [2, 4, 8, 16, 32] as the dividend; dividing the

> dividend by the divisor gives the [amount of] chi. If [the amount in] *chi* is not even [i.e. it is a fraction], multiply by 10 and express the fraction in *cun*. If [the amount in] *cun* is not even, use the divisor to determine the fraction [remaining in *cun*]. (Dauben, "Suan shu shu" 117-118)

Problem 3.4 is written in the *Nine Chapters* as follows:

> Now given a skillful weaver, who doubles her product every day. In 5 days she produces [a cloth of] 5 *chi*. Tell: how much does she weave in each successive day?
>
> Answer: On the first day she weaves $1\frac{19}{31}$ *cun*; on the second day, $3\frac{7}{31}$ *cun*; on the third day, $6\frac{14}{31}$ *cun*; on the fourth day, $12\frac{28}{31}$ *cun*; and on the fifth day $25\frac{25}{31}$ *cun*.
>
> Method: Lay down the rates for distribution: 1, 2, 4, 8 and 16. Take their sum as divisor. Multiply 5 *chi* by each rate as dividend. Divide, giving the number of *chi*. (Kangshen et al. 162)

In both the *Suan shu shu* and the *Nine Chapters*, it is said that the woman "doubles her product" every day (Kangshen et al. 162) – although according to Dauben, the character for "doubles" is different in both texts; the *Suan shu shu* uses the character for *zai*, which in this context means "twice as much" (Dauben, "Suan shu shu" 117). The *Nine Chapters*, on the other hand, uses the character for *bei*, which literally "means 'to double'" (Dauben, "Suan shu shu" 118). These problems, however, do share some characters; Dauben reveals that in order to say that the woman "is displeased with herself," the characters *wu zi xi* are used in the *Suan shu shu*, with *wu* meaning "hate, detest, loathe," and *xi* meaning "'happy' or 'pleased'" (Dauben, "Suan shu shu" 117). The *Nine Chapters* uses the characters *zi xi* (and not *wu*) in order to say that the woman is "pleased with herself" (Dauben, "Suan shu shu"

118). (Dauben clarifies that “Shen, Crossley, and Lun translate [*zi xi*] as ‘skillful weaver’” (Dauben, “Suan shu shu” 118).)

In both problems, the reader is told that 5 *chi* is woven by the woman in five days, and is asked how much the woman weaves on each of the five days. The answer in both problems is the same, except that the answer in the *Nine Chapters* is in the lowest terms while the answer in the *Suan shu shu* is not. This is because, as Dauben notes, “the rates for distribution” (Kangshen et al. 162) begin with 1 in the *Nine Chapters* and 2 in the *Suan shu shu*. The *Suan shu shu*’s method is more expansive in that it includes instructions on how to make a fractional or mixed number answer an integral answer by converting *chi* to *cun*. The *Suan shu shu* also advises in its method how to turn a fractional answer into a mixed number. These extra steps are only implied in the *Nine Chapters*, for the answers are given as mixed numbers in *cun*. For this reason, these problems can be seen as a rare case where the *Suan shu shu*’s method is clearer than that of the *Nine Chapters*, but it can be argued that the method in the *Nine Chapters* has an advancement over that of the *Suan shu shu* in that the answer is reduced to the lowest terms.

It is likely that problem 14 of the *Suan shu shu* was reconstructed in the *Nine Chapters* as problem 3.4. The authors of the *Nine Chapters* may have taken this problem from the *Suan shu shu* and slightly modified it so that the answer was in simplest terms, leaving out the supplemental information in the exposition of the problem and the extra steps in the method in order to make the method more concise.

2.4 Chapter 4 of the *Nine Chapters on the Mathematical Art*: Short Width (*Shaoguang*)

This chapter of the *Nine Chapters* is named “Short Width” (*Shaoguang*), which is the name of problem 66 in the *Suan shu shu*. According to Kangshen et al., *shaoguang* refers to the "shorter side of a rectangle," with *guang* meaning "literally width" (Kangshen et al. 196). Kangshen et al. tell us that

> [i]n the first 11 problems of the Chapter the area of a rectangular field is given as 1 *mu* (that is to say, 240 square *bu*) and the length of the field is to be found, whereas the width, shorter than the length, is given as the sum of 1 *bu*, and a series of unit fractions, hence the name of the Chapter. (Kangshen et al. 196)

Thus, one is to find the length of a rectangular field given the area and a width that gradually increases. Crossley et al. also tell us that "[t]his Chapter exhibits working backwards to find a side or diameter, etc. from a known area or volume" (Crossley et. al), particularly with rectangular fields, circles, and spheres.

Chapter 4 of the *Nine Chapters* also provides methods for approximating square roots and cube roots. The method for finding the square root is written (with some notation for footnotes) following problems 12 through 16. (Two unfamiliar terms are used in this rule. They are *shi*, which in this context means dividend; and *fa*, which refers to the divisor.):

The Rule for Extracting the Square Root

> Lay down the given area as *shi*. Borrow a counting rod to determine the digital place. Set it under the unit place of the *shi*.[(i)] Advance [to the left] every two digital places as one step.[(ii)] Estimate the first digit of the root.[(iii)] The estimated number multiplied by the borrowed rod is regarded as *fa*.[(iv)] Then carry out the subtraction.[(v)] After that, double the *fa* as determined *fa*.[(vi)] Prepare the second subtraction. Move the determined *fa* one digit [to the right] and set the borrowed rod as before.[(vii)] Estimate the second digit of the root.[(viii)] Multiply it by the borrowed rod and subjoin the product to the determined *fa*.[(ix)] Then [carry out] the second subtraction.[(x)] Subjoin to the determined *fa* a second time.[(xi)] Proceed

with the operation in the same manner.$^{(xii)}$

If there is a remainder, [the number] is called unextractable, it should be defined as the side on which the square has the area of the *shi*. If the *shi* has a fractional part, reduce it to [an] improper [fraction]. Extract the square root of both the numerator and denominator, which are considered as dividend and divisor respectively, and divide. If the denominator is unextractable, multiply the numerator by the denominator. Extract the root of the product, then divide the root by the denominator. (Kangshen et al. 204)

Crossley et al. explain the *Nine Chapters*' process of finding the square root in their commentary. They do this according to problem 4.12, which asks for "the side of the square" that has "an area 55 225 [square] *bu*" (Crossley et. al 203). (Each step is labeled according to its explanation in the original general rule for finding the square root.):

(i) $x^2 = S = 55\ 225$

(ii) To estimate the first digit of the root, we transform the equation by diminishing the root to $x_1 = \frac{x}{100}$, then $10\ 000{x_1}^2 = 55\ 225$.

(iii) Estimate, $\lfloor x_1 \rfloor = 2$ where $\lfloor x_1 \rfloor$ means the integer part of x_1.

(iv)-(vi) Subtract the root, $y = x_1 - \lfloor x_1 \rfloor = x_1 - 2$ and the equation becomes $10\ 000(y+2)^2 = 55\ 225$, that is $10\ 000y^2 + 40\ 000y = 15\ 225$.

(vii) Now increase the root, $y_1 = 10y$, we have $100{y_1}^2 + 4000y = 15\ 225$.

(viii) Estimate, $\lfloor y_1 \rfloor = 3$.

(ix)-(x) Subtract the root, $z = y_1 - \lfloor y_1 \rfloor = y_1 - 3$, giving $100(z+3)^2 + 4000(z + 3) = 15\ 225$.

(xi) $100z^2 + 4600z = 2325$.

(xii) Increase the root, $z_1 = 10z$, giving ${z_1}^2 + 460z_1 = 2325$. Estimate, $\lfloor z_1 \rfloor = 5$. Now $z_1 = 5$ exactly, so finally we have

$$z = 100\lfloor x_1 \rfloor + 10\lfloor y_1 \rfloor + \lfloor z_1 \rfloor = 235.$$

In the *Suan shu shu*, problem 53 finds the square root by finding the dimensions of a square field with an area of 1 *mu*:

> [Given] a field of 1 *mu*, how many [square] bu are there? [The answer] says: 15 $\frac{15}{31}$ [square] *bu*. The method says: a square 15 *bu* [on each side] is deficient [in area] by 15 [square] *bu*; a square 16 *bu* [on each side] is in excess [in area] by 16 [square] *bu*. [The method] says: combine the excess and deficiency as the divisor; [taking] the deficiency numerator multiplied by the excess denominator and the excess numerator times the deficiency denominator, combine them as the dividend. Repeat this, as in the "method of finding the width." [The "method of finding the width" is found in problem 64, but as Dauben states in a footnote to problem 53, the method being referred to is actually in problem 65.] (Dauben, "Suan shu shu" 152)

As in the *Nine Chapters*, the method for excess and deficiency is used in the *Suan shu shu*. (In the *Nine Chapters*, this is seen in step (iii), where $\lfloor x_1 \rfloor$ is estimated to be 2 because if $\lfloor x_1 \rfloor = 3$, then $10\,000{x_1}^2$ will be larger than 55 225, and if $\lfloor x_1 \rfloor = 1$, then $10\,000{x_1}^2$ will be far too small.) It is determined that the side of the square field must be larger than 15 *bu*, because $(15)^2$ is 15 *bu* less than the area of the field; the side of the field must also be smaller than 16 *bu*, because $(16)^2$ is 16 *bu* more than the area of the field. The *Nine Chapters*, however, differs in its method in that it "allows determination of the square root by an iterative procedure based upon successive approximations by completing squares" (Dauben, "Suan shu shu" 153). This process is a "highly developed...algorithm" compared to the "'excess and deficiency' method" presented in the *Suan shu shu* (Dauben, "Suan shu shu" 153). After determining the excess and deficiency for 16 and 15 respectively, the *Suan shu shu* instructs to "combine the excess and deficiency as the divisor", and then multiply the "deficiency numerator" (which, according to Dauben, means the original speculation of 15 *bu* as the square's side, when the numbers are laid out on the counting board) by the "excess denominator" (meaning the excess that is produced when 16 *bu* is estimated as the side of the square) (Dauben, "Suan shu shu" 152). One is then instructed to multiply "the excess numerator" (the guess of 16 *bu* for the side of the square) by "the deficiency denominator" (the deficiency of 15 when 15 *bu* is used as the estimation of the side of the square). The method in the *Suan shu shu* relies more on trial and error, which becomes more difficult with larger numbers. On the other hand, the algorithm in the *Nine Chapters* makes larger numbers easier to work with. The general rule in the *Nine Chapters* is also more advanced than that of the *Suan shu shu* in that it instructs one on how to handle irrational roots, even going as far as instructing the reader to rationalize the answer after turning it into an improper fraction and then take the square root of the numerator. The differences between the *Suan shu shu* and the *Nine Chapters* in their methods for finding the square root show the advancement of the *Nine Chapters* in its mathematics over the *Suan shu shu*.

2.5 Chapter 5 of the *Nine Chapters on the Mathematical Art*: Construction Consultations (*Shanggong*)

Many problems in Chapter 5 of the *Nine Chapters* calculate the "[v]olumes of different solids," as do several problems in the *Suan shu shu* (Kangshen et al. 251). The solids explored in both texts are the *xianchu*, the *chumeng*, the *chutong* (whose formula for volume is the same as that of the *quchi*, the *pangchi*, and the *minggu*), the cone, the frustum of a cone, and the cylinder. Solids studied in the *Nine Chapters* but not in the *Suan shu shu* are the "'cuboid (i.e. rectangular parallelepiped)..., prism of constant trapezoidal section"

(Kangshen et al. 251), the frustum of a pyramid, the pyramid, the *qiandu* ("right triangular prism" (Kangshen et al. 251)), the *yang-ma*, the *bie 'nao*, the *panchi*, the *minggu*, and the *quchi*. The inclusion of these in the *Nine Chapters* shows the advancement of the *Nine Chapters* over the *Suan shu shu*, and the wider extent of the material within the *Nine Chapters*.

2.6 Chapter 6 of the *Nine Chapters on the Mathematical Art*: Fair Levies (*Junshu*)

This chapter of the *Nine Chapters* contains problems that "provide a further development of the ratio and proportion theory of Chapters 2 and 3" (Kangshen et al. 307).

Problem 6.27 of the *Nine Chapters* is somewhat similar to problem 16 of the *Suan shu shu*. Problem 6.27 examines continued proportions:

> Now given a person carrying cereal through three passes. At the outer pass, one-third is taken away as tax. At the middle pass, one-fifth is taken away. At the inner pass one-seventh is taken away. Assume the remaining cereal is 5 *dou*. Tell: how much cereal is carried originally?
>
> Answer: 10 *dou* $9\frac{3}{8}$ *sheng*.
>
> Method: Lay down 5 *dou*; multiply by the tax numbers 3, 5, 7 successively as dividend. Take the continued product of the remainders 2, 4, 6 as divisor. Divide, giving the number of *dou* to be found. (Kangshen et al. 345)

Problem 16 in the *Suan shu shu* also follows a person who "passes through...three custom houses" (Dauben, "Suan shu shu" 121):

> ***Fu mi*: Carrying Husked Grain**

> A person carrying husked grain not knowing the amount passes through customs; [there are] three customs houses where the tax is $\frac{1}{[3]}$ [*dou* of grain] each time, after which the remaining grain is 1 *dou*. How much grain was being carried at the beginning of the trip? The result says: the [amount of] grain carried [at the beginning of the trip] was 3 *dou* 3 and $\frac{3}{4}$ *sheng*. The method says: put down [on the counting board the amount] for one customs house 2 [*bei*] 3 times as divisor; again put down [on the counting board the amount of] grain of 1 *dou*, multiplying it by 10, again tripling the number of customs houses as the dividend. (Dauben, "Suan shu shu" 121)

Like in the *Nine Chapters*, the *Suan shu shu* shows a person passing through three customs posts (called "custom houses" in the *Suan shu shu* and "passes" in the *Nine Chapters*), and asks how much grain the person had before he started (based on how much the person was taxed at each customs post). The problems differ, however, in that the tax stays the same among the three customs posts in the *Suan shu shu*, while in the *Nine Chapters*, the tax progressively decreases from one customs post to the next. The *Nine Chapters* thus takes a concept from the *Suan shu shu* – proportions in relation to taxation – and presents a more advanced version. Since three customs posts are described in both texts, and since the commodity being taxed in both problems is a grain, it is possible that the authors of the *Nine Chapters* may have adapted their problem from the *Suan shu shu*.

2.7 Chapter 7 of the *Nine Chapters on the Mathematical Art*: Excess and Deficit (*Ying Buzu*)

Chapter 7 of the *Nine Chapters* is devoted to excess and deficiency problems. These problems seek to solve linear equations using "[t]he Excess and Deficit Rule," or what the *Nine Chapters* calls "[t]he Alternative Rule" (Kangshen et al. 359). These rules will be discussed in further detail. Together, these rules form "[t]he Rule of Double False Position" (Kangshen et al. 349).

Problem 7.1 of the *Nine Chapters* is written as follows:

> Now an item is purchased jointly; everyone contributes 8, the excess is 3; everyone contributes 7, the deficit is 4. Tell: the number of people, the item price, what is each?
>
> Answer: 7 people, item price 53.

The method for problems 1 through 4 of this chapter is written as follows:

> The Excess and Deficit Rule
>
> Display the contribution rates; lay down the [corresponding] excess and deficit below. Cross-multiply by the contribution rates; combine them as dividend; combine the excess and deficit as divisor. Divide the dividend by the divisor. [If] there are fractions, reduce them.
>
> To relate the excess and the deficit for the articles jointly purchased: lay down the contribution rates. Subtract the smaller from the greater, take the remainder to reduce the divisor and the dividend. The [reduced] dividend is the price of an item. The [reduced] divisor is the number of people. (Kangshen et al. 359)

Another rule for excess and deficiency is written immediately after:

> The Alternative Rule
>
> Combine the excess and deficit as dividend. Take the contribution rates; subtract the smaller from the greater, the excess is the divisor. Divide the dividend by the divisor obtaining the number of people. Multiply it by the contribution rates; subtract the excess, [or] add the deficit for the item price. (Kangshen et al. 359)

Problems 51 through 53 are the only excess and deficiency problems in the *Suan shu shu*. Problem 51 is written as follows and can be compared to problem 7.1 of the *Nine Chapters*:

> ***Fen qian*: Dividing Coins**
>
> [When] dividing coins, if each person gets 2 coins, then there are three coins too many; if each person gets 3 coins, then there are 2 coins too few. How many people and how many coins are there? The answer says: 5 people and 13 coins. Cross-multiply the excess and deficiency by the denominators [and combine the products] as the dividend, add the numerators together as the divisor; if there is always an excess or likewise a deficiency, cross-multiply the numerators and denominators and put each down (on the counting board) separately; subtract the smaller numerator from the larger numerator; the remainder is the divisor; use the deficiency as the dividend. (Dauben, "Suan shu shu" 150)

The Excess and Deficit Rule is similar to the method used in problem 51 of the *Suan shu shu*; in the *Nine Chapters*, "the contribution rates" of 8 and 7 are placed next to each other on the counting board, while the "[corresponding] excess and deficit [are placed] below [them]" (Kangshen et al. 359). (In the *Suan shu shu*, it is the opposite, with the "contribution rates" of 2 and 3 below their "[corresponding] excess and deficit".) Then, cross-multiplication occurs. The sum of the products is the item price in the case of the *Nine Chapters*, and the number of coins in the case of the *Suan shu shu*; the sum of the excess and deficit is the number of people. Thus, the rules are the same, except that the excess and deficiency are above the contribution rates on the counting board in the *Suan shu shu*, and the excess and deficiency are below the contribution rates in the *Nine Chapters*. This small difference in the rules suggests that the Excess and Deficit Rule may have been taken directly from the *Suan shu shu* and slightly modified.

The *Nine Chapters* provides a more complicated method for solving problems like problem 7.1 of the *Nine Chapters* and problem 51 of the *Suan shu shu*. Kangshen et al. describes this using the formula

$$x = \frac{c_1+c_2}{a_1-a_2},$$

where x is the number of people, c_1 is the excess, c_2 is the deficiency, a_1 is the contribution rate for the excess, and a_2 is the contribution rate for the deficiency. The *Nine Chapters* also uses the formula

$$y = \frac{a_2c_2+a_2c_1}{a_1-a_2},$$

where y is the item price. This additional and more complicated formula shows the advancement of the *Nine Chapters* in excess and deficiency problems over the *Suan shu shu*.

2.8 Chapter 8 of the *Nine Chapters on the Mathematical Art*: Rectangular Arrays (*Fangcheng*)

This chapter is dedicated to "systems of linear equations" (Kangshen et al. 386). In the method for solving these equations, "[t]he coefficients are laid out in an array and then the coefficient matrix of the array is reduced to triangular form using elementary matrix operations" (Kangshen et al. 386). Dauben notes that this method allows one to solve for up to 5 unknowns (as is done in problem 18 of this chapter) (Dauben, "Suan shu shu" 132). Chapter 8 also presents a method on how to treat negative numbers that result from these systems of linear equations. Matrices and negative numbers are neither mentioned nor utilized in the *Suan shu shu*, and were not even known to the Chinese mathematics community when the *Suan shu shu* was published. (Jean-Claude Martzloff accounts that "the notion of negative numbers is to be found...towards the beginning of the first millennium AD, under the Han Dynasty", with its first appearance seemingly in the *Nine Chapters*, though "negative numbers are never found in the statements of problems" (Martzloff 200). It seems that solutions to systems of linear equations with "several unknowns" were also not able to be found before the *Nine Chapters* (Martzloff 249, 250).) For this reason, none of the problems in Chapter 8 of the *Nine Chapters* have counterparts in the *Suan shu shu*. The fact that none of the complex concepts in this chapter are found within the *Suan shu shu*, and were not even known at the time the *Suan shu shu* was written, shows the advanced nature of the mathematics within the *Nine Chapters*. These topics were clearly not taken from the *Suan shu shu* by the authors of the *Nine Chapters*; the authors of the *Nine Chapters* chose to add to the material in the *Suan shu shu* when compiling the *Nine Chapters* by including this – at the time – recent discovery.

2.9 Chapter 9 of the *Nine Chapters on the Mathematical Art*: Right-angled Triangles (*Gougu*)

This chapter examines "a variety of applications of the *gou-gu* relation," with *gou-gu* referring to right triangles (Dauben, "Suan shu shu" 97). The topics studied in this chapter are the *Gougu* Rule (which, "in the West", would be called the Pythagorean Theorem (Kangshen et al. 439)), "[p]roportional relationships between the sides of similar right-angled triangles" (Kangshen et al. 440), and "[*g*]*ougu* numbers (Pythagorean triples)" (Kangshen et al. 440). Right triangles and their applications are not discussed in the *Suan shu shu*. As with the material in Chapter 8 of the *Suan shu shu*, the material within this chapter was discovered after the *Suan shu shu* and included in the *Nine Chapters* by its authors. The inclusion of right triangles and their properties in the *Nine Chapters* shows the advancement of the *Nine Chapters* over the *Suan shu shu* and the authors' desire to "[supplement]" the *Suan shu shu* (Kangshen et al. 53).

3 Conclusions

Liu Hui's account of the compilation of the *Nine Chapters* appears to contain some truth; after the *Nine Arithmetical Arts* was burned, the authors of the *Nine Chapters* seem to have used "the remains of incomplete old manuscripts" of the *Nine Arithmetical Arts* in order to reconstruct the *Nine Chapters* (Kangshen et al. 53). However, the authors did not compile the *Nine Chapters* using only portions of the *Nine Arithmetical Arts*, but also parts of the *Suan shu shu*. This would explain the nearly identical problems between the two texts, such as problem 14 in the *Suan shu shu* compared to problem 3.4 of the *Nine Chapters*. Liu Hui mentions that the authors of the *Nine Chapters* "revised and supplemented [the *Nine Arithmetical Arts*]" in their text (Kangshen et al. 53). The authors of the *Nine Chapters* did this with the *Suan shu shu* as well, revising the *Suan shu shu* by providing clearer, more advanced explanations and creating a better organization of the text. (The superior organization of the *Nine Chapters* over that of the *Suan shu shu* is shown especially by the classification of problems into chapters in the *Nine Chapters* and the consistency among the format of problems.) The *Nine Chapters* supplements the material of the *Suan shu shu* by including more problems on topics explored in the *Suan shu shu* (such as excess and deficiency), and introducing topics that were not found in the *Suan shu shu* (including systems of linear equations that have more than two unknowns). Thus, the *Nine Chapters* shows itself to be more extensive and advanced than the *Suan shu shu*, but also reveals that it has a foundation in the *Suan shu shu*. While those who compiled the *Nine Chapters* most likely used at least one other reference (the *Nine Arithmetical Arts* being one of them) in their reconstruction of the *Nine Chapters*, it must be considered that the *Suan shu shu* was a resource used in the compilation of the *Nine Chapters*.

The scholar Christopher Cullen offers an explanation that supports the conclusions reached in this paper. Cullen's view is that like medical books, the *Nine Chapters* was compiled from "textlets." A textlet is "a shorter piece of writing capable of being transmitted on its own" (Cullen). Books can be formed by compiling these textlets that were often collected by academics. (Often, students would receive textlets from their teachers.) A textlet, while never exactly the same as another textlet, might contain information similar to other textlets that would permit one to compile a book based on this similar material. Cullen believes that the *Suan shu shu* was created from such textlets. Textlets from or similar to the *Suan shu shu* could have circulated to those who compiled the *Nine Chapters*, thus explaining some of the correlations between the two texts. Cullen's theory fits with Liu Hui's account that the *Nine Chapters* was compiled from "incomplete old manuscripts" (Kangshen et al. 53), for these manuscripts could have included textlets from the *Suan shu shu*.

If the authors of the *Nine Chapters* did truly look to the *Suan shu shu* as a guide, this would have considerable implications for the *Suan shu shu*; as the *Nine Chapters* is considered the most influential mathematical text from ancient China, if a significant amount of material from the *Nine Chapters* is taken from the *Suan shu shu*, the *Suan shu shu* would be looked at as an almost prototype for the *Nine Chapters*. This would give the *Suan shu shu* acclaim in the mathematics community that it has not yet earned, as it is often overshadowed by other more advanced ancient Chinese texts that have been in the public domain for a longer period of time. Hopefully the mathematics community will come to see the impact that the *Suan shu shu* has had on the *Nine Chapters*, and will give the *Suan shu shu* the recognition it merits.

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